DEPTH ZERO SUPERCUSPIDAL L-PACKETS FOR INNER FORMS OF GSp_4

JAIME LUST

Abstract. We show that for any tame regular discrete series parameter of GSp_4 or its inner form $GU_2(D)$, the L-packet attached by the local Langlands conjecture [GT], [GTan] agrees with the L-packet of depth zero supercuspidal representations constructed by DeBacker and Reeder [DR].

1. Introduction

Let G be a linear reductive group over a non-archimedean local field k of characteristic 0. The local Langlands conjectures predict that irreducible smooth representations of G should be parametrized by admissible homomorphisms

$$\phi : W_k' \longrightarrow {}^LG,$$

of the Weil-Deligne group W_k' to the Langlands dual group ${}^LG$. The fiber over such a L-parameter ϕ is a finite set of irreducible smooth representation of G called a L-packet. Such a map should satisfy certain desired properties that characterize the map uniquely such as the preservation of local factors attached to both sides of the correspondence.

Around 2000, Harris and Taylor [HT], and separately Henniart [He2], proved the local Langlands conjecture for GL_n. Cases for small n were established earlier by Kutzko and Henniart. Rogawski [Ro] proved the conjecture for U_2 and U_3. In 2007, Gan and Takeda [GT] proved the local Langlands conjecture for GSp_4. Also, Gan and Tantono [GTan] proved the local Langlands conjecture for $GU_2(D)$, the inner form of GSp_4.

For supercuspidal representations, there is another conjectural classification which is independent of the local Langlands conjectures. It is conjectured that any irreducible supercuspidal representation π of G is of the form

$$\pi \cong \mathrm{c} - \mathrm{Ind}_K^G \sigma,$$

where K is an open compact mod center subgroup of G and σ is a representation of K. Recently much work has been done in this direction. In 1993 Bushnell and Kutzko [BK1] showed that any irreducible supercuspidal representation of GL_n is compactly induced. In 2001, J.K. Yu [Y] constructed a family (K, σ) for any connected reductive group G. In 2007, J.-L. Kim [Ki] proved that, for p large, the family constructed by Yu exhausts all supercuspidal representations of G. Independently, for $p \neq 2$, Stevens [St] constructed another family (K, σ) for the classical groups U_n, Sp_{2n} or SO_n. He also proved that for these groups G, his family exhausts all supercuspidal representations of G.

It is not obvious how to relate the construction of supercuspidal representations via compact induction to the classification given by the local Langlands correspondence. There is a series of papers by Bushnell and Henniart [BH1], [BH2] devoted to answering this question for GL_n. In

the tame case for GL_n, these results were established earlier by Henniart [He1]. The purpose of this paper is towards answering this question for GSp_4.

For pure inner forms of unramified p-adic groups, DeBacker and Reeder [DR] give a parametrization of tame regular discrete series Langlands parameters. For any such parameter they attach a L-packet of compactly induced depth zero supercuspidal representations. For the groups SO_{2n+1} and Sp_{2n}, Savin [Sa] considered the case of the generic depth zero supercuspidal representation attached to a tame regular discrete series parameter. He showed that the parametrization given by DeBacker and Reeder agrees with the lifting of generic supercuspidal representations to a general linear group as given by Jiang and Soudry [JS] in the case of SO_{2n+1}, and by Cogdell, Kim, Piatetski-Shapiro, and Shahidi [CKPS] in the case of Sp_{2n}. We follow the general strategy for proof as in [Sa], however we consider the entire L-packet attached to a parameter, which also covers the case of non-generic representations.

We consider the subset of tame regular discrete series parameters of GSp_4. The construction of DeBacker and Reeder applies to GSp_4 but not to $GU_2(D)$. We extend their construction to give L-packets of depth zero supercuspidal representations for both GSp_4 and $GU_2(D)$, that agree with DeBacker and Reeder for GSp_4. We note that recently Kaletha [Ka], by an alternate construction based on work of Kottwitz on isocrystals, has extended the work of DeBacker and Reeder to non-pure inner forms of an unramified group G.

Our main result is that the L-packets given by DeBacker and Reeder agree with those given by the local Langlands conjectures for GSp_4 and $GU_2(D)$:

Theorem 1.1. *Let ϕ be a tame regular discrete series L-parameter. Let L_ϕ^{DR} be the L-packet of depth zero supercuspidal representations of $GSp_4(k)$ or $GU_2(D)$ corresponding to ϕ by the construction of DeBacker and Reeder given in Section 4.5. Let L_ϕ^{GT} be the L-packet of supercuspidal representations of $GSp_4(k)$ or $GU_2(D)$ corresponding to ϕ via the local Langlands conjecture for GSp_4 or $GU_2(D)$. Then*

$$L_\phi^{DR} = L_\phi^{GT}.$$

In [GT] and [GTan], the local Langlands classification is characterized by the preservation of L-factors and ϵ-factors. To prove Theorem 1.1, we need to show that certain L-functions have poles at $s = 0$. By theory of Shahidi [Sh], this question is equivalent to studying the reducibility points of the generalized principal series

$$I(s, \pi \boxtimes \sigma) = \mathrm{Ind}_P^G \delta_P^{1/2} \pi \boxtimes \sigma |\det|^s.$$

Here, π is an irreducible representation of $GSpin_5(k) \cong GSp_4(k)$ or $GSpin_{4,1}(k) \cong GU_2(D)$, and σ a representation of $GL_{2m}(k)$, where $G = GSpin_{4m+5}(k)$ or $GSpin_{2m+4,2m+1}(k)$, and $P = M \cdot N$ is the parabolic with Levi factor $M = GSpin_5(k) \times GL_{2m}(k)$ or $GSpin_{4,1}(k) \times GL_{2m}(k)$.

We need to show:

Theorem 1.2. *Let π be a depth zero supercuspidal representation of $GSpin_5(k)$ or $GSpin_{4,1}(k)$ corresponding to a tame regular discrete series L-parameter $\phi = \phi_1 \oplus \cdots \oplus \phi_r$, $r = 1, 2$, by the construction of DeBacker and Reeder given in Section 4.5. Let $\sigma \cong \sigma_{\phi_i}$, where $1 \le i \le r$, be the depth zero supercuspidal representation of $GL_{2m}(k)$ attached to the L-parameter ϕ_i via the local Langlands correspondence for GL_{2m}. Then the generalized principal series $I(s, \pi \boxtimes \sigma)$ reduces at a unique $s_0 > 0$.*

The proof of Theorem 1.2 is achieved by studying a Hecke algebra $\mathcal{H}(G, \rho)$ associated to this family of induced representations. Using the theory of types and covers developed

by Bushnell and Kutzko [BK2], we find a parahoric subgroup $\mathcal{P}$ of G and representation ρ of $\mathcal{P}$ such that irreducible representations in the Bernstein component of $I(s, \pi \boxtimes \sigma)$ are parametrized by simple $\mathcal{H}(G, \rho)$-modules. We use the method of Kutzko and Morris [KM] to give a presentation for $\mathcal{H}(G, \rho)$ and compute its parameters. As $c - \mathrm{Ind}_{\mathcal{P}}^{G}\rho$ is a depth zero representation, using Morris [Mo2] we describe explicit generators and relations for the Hecke algebra

$$\mathcal{H}(G, \rho) \cong \mathrm{End}_G(c - \mathrm{Ind}_{\mathcal{P}}^{G}\rho).$$

We have $\mathcal{H}(G, \rho) = \langle T_0, T_1, T_2 \rangle$, subject to the relations

$$T_0\, T_i = T_i\, T_0, \quad T_i^2 = (p_i - 1)T_i + p_i, \quad i = 1, 2.$$

We see $\mathcal{H}(G, \rho)$ is a Hecke algebra of type $\tilde{A}_1$ tensored with a polynomial algebra. For T_i, $i = 1, 2$, we show there is a subalgebra of $\mathcal{H}(G, \rho)$ which is canonically isomorphic to the endomorphism ring of an induced representation over a finite field, with T_i identified as the unique non-identity generator. In this way computation of the parameters p_i reduce to computations over a finite field, which can be done following a theorem of Lusztig [Lu]. We show the parameters $p_i, i = 1, 2$, of the Hecke algebra $\mathcal{H}(G, \rho)$ are unequal. Theorem 1.2 then follows using results of Matsumoto [Ma] on the Plancherel measure for Hecke algebras of type $\tilde{A}_1$ and work of Harish-Chandra [Sil] on points of reducibility of principal series.

The organization of this paper is as follows. We begin by giving a short description of general symplectic groups and general spin groups in Section 2. In Section 3, we give a complete characterization of tame regular discrete series Langlands parameters for GSp_{2n}. In Section 4 we define the L-packets L_{ϕ}^{GT} and L_{ϕ}^{DR} for GSp_4 and $GU_2(D)$ attached to a tame regular discrete series parameter ϕ. Properties of the local Langlands conjecture for GSp_4 and $GU_2(D)$, proved in [GT] and [GTan] respectively, are described in Section 4.1 and Section 8. In the situation where a theory of L-factors and ϵ-factors has not been fully developed, such as in the case of non-generic supercuspidal representations for GSp_4, the local Langlands correspondence is characterized by the preservation the coarser invariant Plancherel measure. In Section 4.2 we briefly review the DeBacker-Reeder construction. As noted above, this construction only applies to pure inner forms of an unramified p-adic group G, so in Section 4.3 we extend their construction to inner forms in the case that after taking K-points, where K is the maximal unramified extension of k, the adjoint quotient

$$j : G(K) \to G_{ad}(K)$$

remains surjective.

In Section 5, we introduce the generalized principal series $I(s, \pi \boxtimes \sigma)$ and describe its Bernstein component. In Sections 6 we give a presentation of the Hecke algebra $\mathcal{H}(G, \rho)$ and compute its parameters p_i using results of Morris and Lusztig. We note that [Lu, 8.6], which explicitly describes the endomorphism algebra of the induced space $\mathrm{Ind}_{\mathsf{MN}}^{\mathsf{G}}(\rho)$ where G is a finite group of Lie type and ρ is an irreducible cuspidal representation of M, is subject to the condition that G has connected center. To apply the theorem, in all cases we embed our groups G into groups G' that have connected center. Then in Sections 7 and 8, we state the main theorem and show that the L-packets L_{ϕ}^{GT} and L_{ϕ}^{DR} agree. Throughout the paper we identify certain connected reductive linear algebraic groups in terms of their root datum. The appendix gives a description of the root datum of these groups.

Acknowledgements: This paper is based on the author's Ph.D. thesis. The author is grateful to her advisor Wee Teck Gan for suggesting this problem and for his encouragement and advice. The author also thanks Philip Kutzko, Lawrence Morris, Gordan Savin, and Jiu-Kang Yu for helpful conversations.

2. General symplectic and general spin groups

Let k be a non-archimedian local field of characteristic 0 with finite residue field $\mathfrak{f}$ of characteristic p. Let $F = k$ or $\mathfrak{f}$.

2.1. **General symplectic groups.** Let $V_1 = Ff_1 \oplus Ff_2$ be the 2 dimensional vector space over F equipped with the non-degenerate alternating bilinear form $\langle\ ,\ \rangle$ given by

$$\langle f_1, f_2\rangle = -\langle f_2, f_1\rangle = 1, \quad \langle f_i, f_i\rangle = 0.$$

Then $V = V_1^{\oplus n}$ is a symplectic space of dimension $2n$ over F. Let

$$GSp_{2n} := GSp(V) = \{g \in GL(V) : \langle gv_1, gv_2\rangle = \lambda(g)\langle v_1, v_2\rangle\}$$

where $\lambda(g) \in F^\times$ is a scalar. The scalar $\lambda(g)$ is multiplicative and $\mathrm{sim} : GSp(V) \longrightarrow F^\times$ where $\mathrm{sim}(g) = \lambda(g)$ is the similitude character of $GSp(V)$. We have

$$Sp(V) = \{g \in GSp(V) : \lambda(g) = 1\}.$$

We now define $GU_{2n}(D)$, an inner form of GSp_{4n}. Let D be the quaternion division algebra over k. Let $V_2 = De_1 \oplus De_2$ be the 2-dimensional vector space over D equipped with a Hermitian form with inner product given by

$$\langle e_1, e_2\rangle = \langle e_2, e_1\rangle = 1, \quad \langle e_i, e_i\rangle = 0.$$

Also,

$$\langle dv_1, d'v_2\rangle = \tau(d)\langle v_1, v_2\rangle d'$$

where τ is the standard involution on D. Then $V = V_2^{\oplus n}$ is a Hermitian space of dimension $2n$ over D and

$$GU_{2n}(D) = \{g \in \mathrm{Aut}_D(V) : \langle gv_1, gv_2\rangle = \mu(g)\langle v_1, v_2\rangle\}$$

where $\mu(g) \in k^\times$ is a scalar.

2.2. **General spin groups.** In this section we define general spin groups $GSpin_m$, and related groups.

Let V be an m dimensional vector space over F equipped with a non-degenerate quadratic form $\langle\ ,\ \rangle$. Then

$$GO(V) = \{g \in GL(V) : \langle gv_1, gv_2\rangle = \lambda(g)\langle v_1, v_2\rangle\}$$

where $\lambda \in F^\times$ is a scalar. The similitude character of $GO(V)$ is $\mathrm{sim} : GO(V) \longrightarrow F^\times$ where $\mathrm{sim}(g) = \lambda(g)$. Let $GSO(V)$ be the connected component of $GO(V)$, as the latter is not a connected algebraic group. We have

$$SO(V) = \{g \in GSO(V) : \lambda(g) = 1\}.$$

Let $C(V) = C^+(V) \oplus C^-(V)$ be the Clifford algebra of V with its 2-grading. There is a canonical embedding $V \hookrightarrow C^-(V)$. Then

$$GSpin(V) = \{g \in C^+(V)^\times \mid gg^\iota = \nu(g), \text{ and } gVg^{-1} = V\}$$

where ι is the main involution of $C(V)$ and $\nu(g)$ is a scalar. The scalar $\nu(g)$ is multiplicative and $\mathrm{sim} : GSpin(V) \longrightarrow F^\times$ where $\mathrm{sim}(g) = \nu(g)$ is the similitude character of $GSpin(V)$. Denote by

$$Spin(V) = \{g \in GSpin(V) : \nu(g) = 1\}.$$

As algebraic groups, we have the exact sequence

$$1 \longrightarrow Z^0 \longrightarrow GSpin(V) \longrightarrow SO(V) \longrightarrow 1,$$

where Z^0 is the connected center of $GSpin(V)$.

Over a p-adic field k, there are two quadratic forms of dimension m and discriminant 1. Let $V^+ = H^{2n} \oplus \langle 1\rangle$ or $V^+ = H^{2n}$ be the split quadratic space of dimension $m = 2n+1$ or $2n$ and discriminant 1. Here $H = ke_1 \oplus ke_2$ is the hyperbolic plane with inner form given by

$$\langle e_1, e_2\rangle = \langle e_2, e_1\rangle = 1, \quad \langle e_i, e_i\rangle = 0.$$

Let $V^- = H^{2(n-1)} \oplus D_0$ be the non-split quadratic space of dimension $2n+1$ and discriminant 1, where D_0 is the subset of the quaternion k-algebra D of elements of reduced trace 0 and the quadratic form on D_0 is given by the reduced norm. Let

$$GSO_m = GSO(V^+), \quad SO_m = SO(V^+).$$

Define

$$GSpin_m = GSpin(V^+),$$

and

$$GSpin_{n+2,n-1} = GSpin(V^-).$$

Over a finite field $\mathfrak{f}$ of characteristic p, let $V^+ = H^{2n} \oplus \langle 1\rangle$ or $V^+ = H^{2n}$ be the quadratic space of dimension $m = 2n+1$ or $2n$ over $\mathfrak{f}$ where $H = \mathfrak{f}e_1 \oplus \mathfrak{f}e_2$ is the hyperbolic plane with inner product given as above. Let

$$GSpin_m = GSpin(V^+).$$

For $\mathfrak{f} = \mathbb{F}_q$, let $V^- = H^{2(n-1)} \oplus \mathbb{F}_{q^2}$ where $\mathbb{F}_{q^2}$ is the unique quadratic extension of $\mathbb{F}_q$ with quadratic form given by $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}$. Let

$${}^2GSpin_{2n} = GSpin(V^-).$$

3. Parameters

We first set some notation. Let k be a non-archimedean local field of characteristic 0 with finite residue field $\mathfrak{f}$ of characteristic p. Let $q = |\mathfrak{f}|$. Let $\bar{k}$ be a fixed algebraic closure of k, K the maximal unramified extension of k in $\bar{k}$, and k_m the unramified extension of degree m of k in $\bar{k}$. Let $\bar{\mathfrak{f}}$ be the residue field of K. Then $\bar{\mathfrak{f}}$ is an algebraic closure of $\mathfrak{f}$.

Let $\mathcal{I}$ be the inertia subgroup of the Galois group $\mathrm{Gal}(\bar{k}/k)$. We have $\mathrm{Gal}(\bar{k}/k)/\mathcal{I} \cong \mathrm{Gal}(K/k) \cong \mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$. A geometric Frobenius Frob is one whose image in $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$ is the automorphism which acts as the inverse of $x \mapsto x^q$ on $\bar{\mathfrak{f}}$. Fix a choice of geometric Frobenius. Then $\langle \mathrm{Im}(\mathrm{Frob})\rangle \subset \mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$ is a dense subgroup and the Weil group is

$$W_k = \mathcal{I} \rtimes \langle \mathrm{Frob}\rangle.$$

The Weil-Deligne group W_k' is defined as $W_k' = W_k \times SL_2(\mathbb{C})$.

Let G be a connected reductive linear algebraic group and T a maximal torus of G. Let $X = X^*(T)$ and $X^\vee = X_*(T)$ be the groups of algebraic characters and cocharacters of T. Let Φ and $\Phi^\vee$ be the sets of roots and coroots of T. The quadruple

$$\Psi = (X, \Phi, X^\vee, \Phi^\vee)$$

is the root datum for G. Up to isomorphism, there is a unique complex reductive group $\hat{G}$ with root datum $(X^\vee, \Phi^\vee, X, \Phi)$ dual to that of G. We have $\hat{G}$ is the Langlands dual group of G.

The dual group of GSp_4 is $GSp_4(\mathbb{C})$. By a Langlands parameter for GSp_4 we mean a continuous homomorphism

$$\phi : W_k' \longrightarrow GSp_4(\mathbb{C})$$

such that $\phi(\mathrm{Frob})$ is semisimple. Such a parameter is said to be admissible, and homomorphisms are taken up to $\hat{G}$ conjugacy.

Definition 3.1. Let ϕ be a Langlands parameter.

(i) We say that ϕ is tame if ϕ is trivial on the wild inertia group $\mathcal{I}^+$, the maximal pro-p subgroup of $\mathcal{I}$.
(ii) We say the ϕ is regular if the centralizer in $\hat{G}$ of $\phi(\mathcal{I})$ is a maximal torus $\hat{T}$ in $\hat{G}$.
(iii) We say that ϕ is discrete series if the identity component of the centralizer in $\hat{G}$ of $\phi(W_k)$ is equal to the identity component of the center $\hat{Z}$ of $\hat{G}$.

For k_{2d} the unramified extension of k of degree $2d$, the Weil group of k_{2d} is

$$W_{k_{2d}} = \langle \mathrm{Frob}^{2d} \rangle \ltimes \mathcal{I}.$$

Lemma 3.2. *Fix $\lambda : W_k \longrightarrow \mathbb{C}^\times$. Let $V(\eta) = \mathrm{Ind}_{W_{k_{2d}}}^{W_k} \eta$, where $\eta : W_{k_{2d}} \longrightarrow \mathbb{C}^\times$ is a continuous character. Then*

(i) *$V(\eta)$ is tame $\iff$ η is trivial on the wild inertia subgroup $\mathcal{I}^+$,*
(ii) *$V(\eta)$ is irreducible $\iff$ $\eta, \eta^{\mathrm{Frob}}, \ldots, \eta^{\mathrm{Frob}^{2d-1}}$ are pairwise distinct,*
(iii) *Under (ii), $V(\eta)$ is symplectic with similitude character $\lambda \iff \eta \cdot \eta^{\mathrm{Frob}^d} \cong \lambda|_{W_{k_{2d}}}$ and $\eta(\mathrm{Frob}^{2d}) = -\lambda(\mathrm{Frob}^d)$.*

Proof. (i) Since $\mathcal{I} \subset W_{k_{2d}}$, by the definition of induced representation $\phi : W_k \longrightarrow GL(V(\eta))$ is trivial on $\mathcal{I}^+$ if and only if η is trivial on $\mathcal{I}^+$.

(ii) A set of coset representatives for $W_{k_{2d}} \setminus W_k \,/\, W_{k_{2d}}$ is $\{\mathrm{Frob}, \mathrm{Frob}^2, \ldots, \mathrm{Frob}^{2d}\}$. Define $\eta^{\mathrm{Frob}^i} : W_{k_{2d}} \longrightarrow \mathbb{C}^\times$ by

$$\eta^{\mathrm{Frob}^i}(w) = \eta((\mathrm{Frob}^i)^{-1} w \mathrm{Frob}^i).$$

By Mackey's irreducibility criterion, $V(\eta)$ is irreducible if and only if

(a) η is irreducible, and
(b) the conjugates of η, namely $\eta, \eta^{\mathrm{Frob}}, \ldots, \eta^{\mathrm{Frob}^{2d-1}}$, are pairwise distinct.

Therefore, since η is 1-dimensional, $V(\eta)$ is irreducible if and only (b) is satisfied.

(iii) Assume (ii) is satisfied. Let λ be a character of W_k. There exists a W_k-equivariant map

$$B : V(\eta) \otimes V(\eta) \longrightarrow \lambda$$

if and only if $V(\eta) \cong V(\eta)^\vee \otimes \lambda$. Since $V(\eta)$ is irreducible, any such nonzero map B would be nondegenerate. Now,

$$V(\eta)^\vee \otimes \lambda \cong V(\eta^\vee) \otimes \lambda \cong V(\eta^{-1}) \otimes \lambda \cong V(\eta^{-1} \cdot \lambda|_{W_{k_{2d}}}),$$

where $\eta^\vee = \eta^{-1}$. By Frobenius reciprocity,

$$\mathrm{Hom}_{W_k}(V(\eta^{-1} \cdot \lambda|_{W_{k_{2d}}}), V(\eta)) = \mathrm{Hom}_{W_{k_{2d}}}(\eta^{-1} \cdot \lambda|_{W_{k_{2d}}}, V(\eta)|_{W_{k_{2d}}}).$$

Also by Frobenius reciprocity

$$V(\eta)|_{W_{k_{2d}}} \cong \eta \oplus \eta^{\mathrm{Frob}} \oplus \cdots \oplus \eta^{\mathrm{Frob}^{2d-1}}.$$

Therefore,

$$V(\eta) \cong V(\eta)^\vee \otimes \lambda \Longleftrightarrow \eta^{-1} \cdot \lambda|_{W_{k_{2d}}} \cong \eta^{\mathrm{Frob}^i} \Longleftrightarrow \lambda|_{W_{k_{2d}}} \cong \eta \cdot \eta^{\mathrm{Frob}^i}$$

for some $0 \leq i \leq 2d-1$.

Assume such a W_k-equivariant map B exists. Then $\lambda|_{W_{k_{2d}}} \cong \eta \cdot \eta^{\mathrm{Frob}^i}$ for some $0 \leq i \leq 2d-1$. Since λ is a character of W_k, for any power l,

$$(\lambda|_{W_{k_{2d}}})^{\mathrm{Frob}^l}(w) = \lambda(\mathrm{Frob}^{-l} w \mathrm{Frob}^l) = \lambda(\mathrm{Frob}^{-l})\lambda(w)\lambda(\mathrm{Frob}^l) = \lambda|_{W_{k_{2d}}}(w).$$

Then

$$\eta \cdot \eta^{\mathrm{Frob}^i} = (\eta \cdot \eta^{\mathrm{Frob}^i})^{\mathrm{Frob}^l} = \eta^{\mathrm{Frob}^l} \cdot \eta^{\mathrm{Frob}^{i+l}}$$

for any power l. Taking $l = i$, it follows $\eta^{\mathrm{Frob}^{2i}} = \eta$. Since $0 \leq i \leq 2d-1$, this is only possible if $i = 0$ or d since $\eta, \eta^{\mathrm{Frob}}, \ldots, \eta^{\mathrm{Frob}^{2d-1}}$ are pairwise distinct. Also, since $\eta, \eta^{\mathrm{Frob}}, \ldots, \eta^{\mathrm{Frob}^{2d-1}}$ are pairwise distinct,

$$\eta \cdot \eta \neq \eta \cdot \eta^{\mathrm{Frob}^d}$$

so that $\lambda|_{W_{k_{2d}}} \cong \eta \cdot \eta^{\mathrm{Frob}^i}$ for $i = 0$ or $i = d$ but not both.

We need to determine under what conditions B is symplectic. By Frobenius reciprocity

$$V(\eta)|_{W_{k_{2d}}} = \bigoplus_{i=0}^{2d-1} \eta^{\mathrm{Frob}^i}.$$

Then

$$B : \left(\bigoplus_{i=0}^{2d-1} \eta^{\mathrm{Frob}^i}\right) \otimes \left(\bigoplus_{i=0}^{2d-1} \eta^{\mathrm{Frob}^i}\right) \longrightarrow \lambda|_{W_{k_{2d}}}$$

is a $W_{k_{2d}}$-equivariant map. With respect to this basis, by restriction,

$$B : \eta^{\mathrm{Frob}^i} \otimes \eta^{\mathrm{Frob}^j} \longrightarrow \lambda|_{W_{k_{2d}}}$$

is a nonzero $W_{k_{2d}}$-invariant map if the entry t_{ij} in the matrix of the form B is nonzero. In the case $\lambda|_{W_{k_{2d}}} \cong \eta \cdot \eta$,

$$t_{ij} \neq 0 \Longleftrightarrow \eta^{\mathrm{Frob}^i} \cdot \eta^{\mathrm{Frob}^j} \cong \lambda|_{W_{k_{2d}}} \Longleftrightarrow i = j.$$

We see in this case B is not symplectic. In the case $\lambda|_{W_{k_{2d}}} \cong \eta \cdot \eta^{\mathrm{Frob}^d}$,

$$t_{ij} \neq 0 \Longleftrightarrow \eta^{\mathrm{Frob}^i} \cdot \eta^{\mathrm{Frob}^j} \cong \lambda|_{W_{k_{2d}}} \Longleftrightarrow |j - i| = d.$$

We can choose eigenvectors in the subspaces η and η^{Frob^d} of $V(\eta)|_{W_{k_{2d}}}$ such that the matrix of $\phi(\mathrm{Frob}^d)$ on $\eta \oplus \eta^{\mathrm{Frob}^d}$ has the form $\begin{pmatrix} 0 & 1 \\ \eta(\mathrm{Frob}^{2d}) & 0 \end{pmatrix}$. This matrix preserves $\begin{pmatrix} 0 & t_{0d} \\ t_{d0} & 0 \end{pmatrix}$ up to λ, where as B is either symplectic or orthogonal, either $t_{0d} = -t_{d0}$ or $t_{0d} = t_{d0}$. Therefore $\begin{pmatrix} 0 & t_{0d} \\ t_{d0} & 0 \end{pmatrix}$ is preserved up to λ if and only if

$$\eta(\mathrm{Frob}^{2d}) = \pm\lambda(\mathrm{Frob}^d).$$

Hence, B is symplectic if and only if $\eta(\mathrm{Frob}^{2d}) = -\lambda(\mathrm{Frob}^d)$.

Assume $\eta \cdot \eta^{\mathrm{Frob}^d} \cong \lambda|_{W_{k_{2d}}}$ and $\eta(\mathrm{Frob}^{2d}) = -\lambda(\mathrm{Frob}^d)$. Extend $\eta \cdot \eta^{\mathrm{Frob}^d}$ to a character of W_k such that the extension is isomorphic to λ. Then there exists a nondegenerate W_k-equivariant map $B : V(\eta) \otimes V(\eta) \longrightarrow \lambda$. By what we have said above B is symplectic. □

Lemma 3.3. *All tame regular discrete series Langlands parameters for $GSp_{2n}(k)$ with similitude character λ are of the form $\phi : W_k \longrightarrow GSp(V)$ such that*

$$V = V(\eta_1) \oplus \cdots \oplus V(\eta_s) \ \textit{ where } \ V(\eta_i) = \mathrm{Ind}_{W_{k_{2d_i}}}^{W_k} \eta_i$$

satisfying the conditions of Lemma 3.2, where k_{2d_i} are unramified extensions of k of degree $2d_i$ such that $d_1 + \cdots + d_s = n$. In addition,

(i) *if $k_{2d_i} = k_{2d_j}$, η_i is not equal to any conjugate $\eta_j^{\mathrm{Frob}^k}, 0 \le k \le d_j$, of η_j so the $V(\eta_i)$ are pairwise non-isomorphic and*
(ii) *the similitude character of $V(\eta_i)$ is λ for all i.*

Proof. Let $\phi : W_k \to GSp(V)$, where V is a $2n$ dimensional complex vector space, be a tame regular discrete series Langlands parameter with similitude character λ. As ϕ is discrete series, the identity component of the centralizer in $GSp(V)$ of $\phi(W_k)$ is equal to the identity component of the center of $GSp(V)$, which implies that V is multiplicity free. We have,

$$V = \bigoplus V_i$$

where any two V_i are pairwise non-isomorphic and each V_i is symplectic with similitude character λ. Therefore, it suffices to show any irreducible tame regular $\phi : W_k \to GSp(V)$ with similitude character λ is of the form $V(\eta)$ with η as in Lemma 3.2.

Let ϕ be such a parameter where $\dim V = 2d$. As ϕ is tame, it factors through the tame inertia group $\mathcal{I}_t = \mathcal{I}/\mathcal{I}^+ \simeq \varprojlim \mathfrak{f}_n^\times$, where $\mathfrak{f}_n$ is the degree n extension of $\mathfrak{f}$ the residue field of k. Then, ϕ factors through $\mathfrak{f}_n^\times$ for some $n \ge 1$ and since $\mathfrak{f}_n^\times$ is cyclic, $\phi(\mathcal{I}_t)$ is cyclic. There is a basis of V such that

$$\phi|_{\mathcal{I}_t} = \bigoplus_i \chi_i,$$

where the χ_i are characters. Since ϕ is regular the centralizer of $\phi(\mathcal{I})$ is a maximal torus, so the χ_i are pairwise distinct. Since V is irreducible, Frob permutes the set $\{\chi_i\}$ transitively. Note that Frob^{2d} induces the trivial permutation on $\{\chi_i\}$. Choose $\eta \in \{\chi_i\}$. Then

$$\mathrm{Stab}_{W_k}\eta = \langle \mathrm{Frob}^{2d} \rangle \ltimes \mathcal{I} = W_{k_{2d}}.$$

By Frobenius reciprocity

$$\dim(\mathrm{Hom}_{W_{k_{2d}}}(\eta, \phi|_{W_{k_{2d}}})) = \dim(\mathrm{Hom}_{W_k}(\mathrm{Ind}_{W_{k_{2d}}}^{W_k} \eta, \phi)).$$

Therefore

$$\phi \cong \mathrm{Ind}_{W_{k_{2d}}}^{W_k} \eta =: V(\eta).$$

We have that $V(\eta)$ is tame, irreducible, and symplectic with similitude character λ, and therefore satisfies the conditions of Lemma 3.2. □

4. Packets

4.1. **Local Langlands for GSp_4 and $GU_2(D)$.** In [GT], Gan and Takeda prove the local Langlands conjecture for GSp_4. They define a surjective finite-one-map

$$L : \Pi(GSp_4) \longrightarrow \Phi(GSp_4)$$

from the set of isomorphism classes of irreducible smooth representations of $GSp_4(k)$ to the set of equivalence classes of admissible homomorphisms $\phi : W_k' \longrightarrow GSp_4(\mathbb{C})$ taken up to $GSp_4(\mathbb{C})$ conjugacy. Let L_ϕ^{GT} be the fiber of L over ϕ. The map L [GT, §1 Main Theorem] satisfies many expected and desired properties, such as preservation of local factors attached to both sides of the correspondence, that determine the map uniquely. We will discuss some of these properties further in section 8. A packet L_ϕ^{GT} is parametrized by the set of irreducible characters of the component group

$$A_\phi = \pi_0(C_{GSp_4(\mathbb{C})}(Im(\phi))).$$

In [GTan], Gan and Tantono extend the local Langlands correspondence for GSp_4 to an analogous result for the inner form $GU_2(D)$. They define a surjective finite-to-one map

$$L : \Pi(GU_2(D)) \longrightarrow \Phi(GU_2(D))$$

from the set of isomorphism classes of irreducible smooth representations of $GU_2(D)$ to the set of relevant L-parameters for $GU_2(D)$. A L-parameter ϕ in $\Phi(GSp_4)$ is said to be relevant for $GU_2(D)$ if it does not factor through any irrelevant parabolic subgroup. Also let L_ϕ^{GT}, the L-packet of representations of $GU_2(D)$, denote the fiber of L over a relevant parameter ϕ. The map L satisfies analogous properties to those for GSp_4 which characterize the map uniquely.

We have the modified component group

$$B_\phi = \pi_0(C_{Sp_4(\mathbb{C})}(Im(\phi))).$$

We have an injection of the group of irreducible characters $\mathrm{Irr}(A_\phi) \hookrightarrow \mathrm{Irr}(B_\phi)$ which identifies $\mathrm{Irr}(A_\phi)$ as the subgroup of characters of B_ϕ which are trivial on the image of the center $Z_{Sp_4(\mathbb{C})}$ of $Sp_4(\mathbb{C})$. A parameter ϕ is relevant for $GU_2(D)$ if and only if $\mathrm{Irr}(B_\phi) \neq \mathrm{Irr}(A_\phi)$ and an L-packet L_ϕ^{GT} for $GU_2(D)$ is naturally parametrized by the set $\mathrm{Irr}(B_\phi) \setminus \mathrm{Irr}(A_\phi)$.

4.2. **DeBacker-Reeder L-packets.** Given a TRD parameter ϕ of an unramified p-adic group G, by explicit construction, DeBacker and Reeder associate an L-packet of depth zero supercuspidal representations distributed among the pure inner forms of G. In the following, for $GSp_4 \cong GSpin_5$ we extend their construction to associate to a TRD parameter ϕ an L-packet $\Pi(\phi)$ of depth zero supercuspidal representations distributed among the inner forms of GSp_4.

The inner forms of a p-adic group G are parametrized by classes in the Galois cohomology set $H^1(k, G_{ad})$. We have that $\mathrm{H}^1(k, PGSp_4) = \{\pm 1\}$ and the group GSp_4 has one inner form, namely $GU_2(D)$. Members of the L-packet $\Pi(\phi)$ corresponding to a parameter ϕ are parametrized by the irreducible characters of the component group $\mathrm{Irr}(B_\phi)$, where

$$B_\phi = \pi_0(C_{\widehat{G_{ad}}}(Im(\phi))).$$

By restriction, any $\rho \in \mathrm{Irr}(B_\phi)$ determines a character on $\pi_0(Z({}^LG_{ad}))$. Then via Kottwitz' isomorphism [Ko] which says $\pi_0(Z({}^LG_{ad})) \simeq H^1(k, G_{ad})$, this character determines a class $\omega_\rho \in H^1(k, G_{ad})$. Using the correspondence $\rho \to \omega_\rho$ we distribute the representations in the L-packet $\Pi(\phi)$ between GSp_4 and $GU_2(D)$. Set

$$\Pi(\phi) = \coprod_{\omega \in H^1(k, G_{ad})} \Pi(\phi, \omega)$$

where

$$\Pi(\phi, \omega) = \{\pi(\phi, \rho) : \rho \in \mathrm{Irr}(B_\phi), \omega_\rho = \omega\}.$$

We now describe the representations $\pi(\phi, \rho)$.

4.3. **DeBacker-Reeder construction.** Here we briefly review the DeBacker and Reeder construction for a quasi-split k-group G such that G is K-split. Denote by $G = G(K)$ the K-rational points of G and $G^F = G(k)$ the k-rational points of G, where F is the Frobenius automorphism of G. Also, identify an $\mathfrak{f}$-group G with its group of $\bar{\mathfrak{f}}$-rational points. Denote $\mathsf{G}^F = \mathsf{G}(\mathfrak{f})$. Let T be a maximal k-torus of G such the T is K-split. The dual group of T is the complex torus $\hat{T} = X \otimes \mathbb{C}^\times$ which is a maximal torus in $\hat{G}$ satisfying

$$X^*(\hat{T}) = X_*(T) = X^\vee, \quad X_*(\hat{T}) = X^*(T) = X.$$

Let

$$\mathcal{A} = X^\vee \otimes \mathbb{R} \subset \mathcal{B}(G)$$

be the apartment in the Bruhat-Tits building of G determined by T. Denote by W the affine Weyl group of T in G, where $W \simeq N_G(T)/{}^0T$ for ${}^0T$ the maximal bounded subgroup of T. For $j : G \to G_{ad}$ the adjoint quotient, let

$$X_{ad}^\vee = X_*(j(T)), \quad \mathcal{A}_{ad} = X_{ad}^\vee \otimes \mathbb{R},$$

and let W_{ad} be the affine Weyl group of $j(T) = T_{ad}$ in G_{ad}. We also write

$$j : X^\vee \to X_{ad}^\vee, \quad j : W \to W_{ad}$$

for the maps induced by j. Denote by ϑ the automorphisms of $X^\vee, X_{ad}^\vee, \mathcal{A}, \mathcal{A}_{ad}, W, W_{ad}$ induced by F. Choose a F-fixed hyperspecial vertex $o \in \mathcal{A}_{ad}$.

Let $\phi : W_k' \to \langle \hat{\vartheta} \rangle \ltimes \hat{G}$ be a TRD parameter for G. As ϕ is regular, $\phi(\mathrm{Frob}) = \hat{\vartheta} f$ where $f \in N_{\hat{G}}(\hat{T})$. Denote by $\hat{w}$ the image of f in $\hat{W}_o = N_{\hat{G}}(\hat{T})/\hat{T}$. For any $\hat{\sigma} \in \mathrm{Aut}(X)$, we define $\sigma \in \mathrm{Aut}(X^\vee)$ by $\langle \eta, \sigma\lambda \rangle = \langle \hat{\sigma}\eta, \lambda \rangle, \eta \in X, \lambda \in X^\vee$. Therefore, $\hat{w}$ induces a dual automorphism w of X, where $w \in W_o = N_G(T)/T$. Given $\lambda \in X^\vee$, let $t_\lambda \in W$ be the corresponding translation given by $x \mapsto \lambda + x$. As an automorphism of $X^\vee$, w is a linear transformation on $\mathcal{A}$, so define

$$\sigma_\lambda = t_\lambda w \vartheta \in W \rtimes \langle \vartheta \rangle.$$

Since ϕ is a discrete series parameter we have $(X_{ad}^\vee)^{w\vartheta} = \{0\}$. The operator $I - w\vartheta$ acts invertibly on $\mathcal{A}_{ad}$, so σ_λ has a unique fixed point there, namely

$$x_\lambda = (I - w\vartheta)^{-1} t_{j\lambda} \cdot o.$$

Denote by $\tilde{x}_\lambda$ the preimage of x_λ in $\mathcal{A}^{\sigma_\lambda}$. By [DR, 4.4.1] we have $\mathcal{A}^{\sigma_\lambda} = \tilde{x}_\lambda$. Let J_λ be the unique facet of $\mathcal{A}$ containing $\tilde{x}_\lambda$, as in [DR, 2.7]. Let K_λ be the parahoric subgroup of G given by J_λ, and $\mathsf{G}_\lambda = K_\lambda / K_\lambda^+$ where K_λ^+ is the pro-unipotent radical of K_λ.

We have that $C_{\hat{G}}(Im(\phi)) = \hat{T}^{\widehat{w\vartheta}}$. Recall $A_\phi = \pi_0(C_{\hat{G}}(Im(\phi)))$. Any $\lambda \in X^\vee$ determines a character $\rho_\lambda \in \mathrm{Irr}(A_\phi)$ by the restriction map $X^\vee \to \mathrm{Hom}(\hat{T}^{\widehat{w\vartheta}}, \mathbb{C}^\times)$ which induces an isomorphism [DR, 4.1]

$$[X^\vee/(1-w\vartheta)X^\vee]_{\mathrm{tor}} \xrightarrow{\sim} \mathrm{Irr}(A_\phi), \quad \lambda \to \rho_\lambda.$$

Let X_w be the preimage in $X^\vee$ of $[X^\vee/(1-w\vartheta)X^\vee]_{\mathrm{tor}}$. For $\lambda \in X_w$, define $[\overline{\lambda}]$ as the image of $[\lambda]$ under the map

$$[X^\vee/(1-w\vartheta)X^\vee]_{\mathrm{tor}} \to [(X^\vee/\mathbb{Z}\Phi^\vee)/(1-\vartheta)(X^\vee/\mathbb{Z}\Phi^\vee)]_{\mathrm{tor}} = \mathrm{Irr}[\pi_0(\hat{Z}^{\hat{\vartheta}})] \simeq H^1(k,G),$$

where the first map is projection and the last map is Kottwitz' isomorphism [DR, 2.4-2.6].

Choose an alcove C_λ in $\mathcal{A}$ that contains J_λ in its closure. There is a unique element $w_\lambda \in W_\lambda$ such that $\sigma_\lambda \cdot C_\lambda = w_\lambda \cdot C_\lambda$, where W_λ is the subgroup of W generated by reflections in the hyperplanes containing J_λ. If we let $y_\lambda = w_\lambda^{-1} t_\lambda w$, then we have two expressions for σ_λ, namely

$$t_\lambda w\vartheta = \sigma_\lambda = w_\lambda y_\lambda \vartheta.$$

By [DR, 2.7], there exists a lift $u_\lambda \in N_G(T)$ of y_λ such that

$$[u_\lambda] = [\overline{\lambda}] \in H^1(k,G).$$

DeBacker and Reeder define

$$F_\lambda = \mathrm{Ad}(u_\lambda) \circ F.$$

This determines the inner form G^{F_λ} on which the representation they construct will live.

Choose a lift $\dot{w}$ of w to an F-stable element of $N_G(T) \cap K_o$ where o is the fixed hyperspecial vertex in $\mathcal{A}_{ad}$. Let $F_w = \mathrm{Ad}(\dot{w}) \circ F$. By [DR, 2.3.1] there exists an element $p_\lambda \in G_\lambda$ such that $p_\lambda \cdot x_\lambda = x_\lambda$ and if $T_\lambda := \mathrm{Ad}(p_\lambda)T$ then $\mathrm{Ad}(p_\lambda)$ intertwines (T, F_w) with (T_λ, F_λ). Since p_λ fixes x_λ we have

$${}^0T_\lambda^{F_\lambda} \subset K_\lambda^{F_\lambda} \subset G^{F_\lambda}.$$

We slightly modify ϕ to obtain a parameter ϕ' of T^{F_w}. The L-group of T^{F_w} is

$${}^L(T^{F_w}) = \langle \hat{w} \rangle \ltimes \hat{T}.$$

We have that the inclusion $\hat{T} \hookrightarrow \hat{G}$ induces a bijection $\hat{T}/(1-\hat{w})\hat{T} \to (\hat{G}/\hat{G}')/(1-\hat{\vartheta})(\hat{G}/\hat{G}')$ where $\hat{G}'$ is the derived group of $\hat{G}$. Define $\phi' : W_k \longrightarrow {}^L(T^{F_w})$ by

$$\phi'(\mathcal{I}) = \phi(\mathcal{I}), \quad \phi'(\mathrm{Frob}) = \hat{w} \ltimes u \in \langle \hat{w} \rangle \ltimes \hat{T},$$

where $u \in \hat{T}$ is any element whose class in $\hat{T}/(1-\hat{w})\hat{T}$ corresponds to the image of f (where $\phi(\mathrm{Frob}) = \hat{\vartheta} \ltimes f$) in $(\hat{G}/\hat{G}')/(1-\hat{\vartheta})(\hat{G}/\hat{G}')$. As given in [DR, 4.3], by the Langlands correspondence for unramified tori, the parameter ϕ' determines a depth-zero character χ_ϕ of T^{F_w}.

We have that ${}^0T_\lambda = T_\lambda \cap G_\lambda$ and that via the reduction mod $\mathfrak{p}$ map, ${}^0T_\lambda$ projects onto T_λ, an F_λ-minisotropic maximal torus in G_λ. Conjugate the character χ_ϕ of T^{F_w} to get a character χ_λ of $T_\lambda^{F_\lambda}$:

$$\chi_\lambda = \chi_\phi \circ \mathrm{Ad}(p_\lambda)^{-1}.$$

The restriction of χ_λ to ${}^0T_\lambda^{F_\lambda}$ factors through a character $\chi_\lambda^0 \in \mathrm{Irr}(\mathsf{T}_\lambda^{F_\lambda})$ which is in general position. By Deligne-Lusztig [DL] induction we have an irreducible cuspidal representation $\epsilon_{\mathsf{T}}\epsilon_{\mathsf{G}}R_{\mathsf{T}_\lambda,\chi_\lambda^0}^{\mathsf{G}_\lambda}$ of $\mathsf{G}_\lambda^{F_\lambda}$. Define

$$\pi_\lambda = c-\mathrm{Ind}_{Z^F K_\lambda^{F_\lambda}}^{G^{F_\lambda}}\big(\chi_\lambda \otimes \epsilon_{\mathsf{T}}\epsilon_{\mathsf{G}}R_{\mathsf{T}_\lambda,\chi_\lambda^0}^{\mathsf{G}_\lambda}\big).$$

By [DR, 4.5.1], $\pi_\lambda \in [\pi(\phi,\rho)]$ is an irreducible supercuspidal representation of G^{F_λ}.

4.4. **Extending DeBacker-Reeder.** We extend the DeBacker-Reeder construction to inner forms of an unramified group G in the situation where $H^1(K,Z)=0$, where Z is the center of G. In this case

$$G(K) \to G_{ad}(K) \to 1$$

is surjective where $j : G \to G_{ad}$ is the adjoint quotient. The notation is the same as in the previous section.

Let ϕ be a TRD parameter for G. We have $\widehat{T_{ad}} = \hat{T} \cap \widehat{G_{ad}}$ is a maximal torus of $\widehat{G_{ad}}$. Since

$$N_{\hat{G}}(\hat{T})/\hat{T} \cong N_{\widehat{G_{ad}}}(\widehat{T_{ad}})/\widehat{T_{ad}}$$

are canonically isomorphic, let $\hat{w}$ also denote the corresponding element in $N_{\widehat{G_{ad}}}(\widehat{T_{ad}})/\widehat{T_{ad}}$. Denote by w the corresponding automorphism of $X_{ad}^\vee$. We apply the DeBacker-Reeder construction as given above to the group G_{ad}. Let $\lambda \in X_{ad}^\vee$. We have a choice of element $\dot{w} \in N_{G_{ad}(k)}(T_{ad}(k)) \cap K_{o,ad}$ where $K_{o,ad}$ is the parahoric subgroup attached to the F-fixed hyperspecial vertex $o \in \mathcal{A}_{ad}$. The DeBacker-Reeder construction also gives us an element $p_\lambda \in G_{ad}(K)$ such that the map $T_{ad} \to \mathrm{Ad}(p_\lambda)T_{ad}$ satisfies $F_\lambda \circ \mathrm{Ad}(p_\lambda) = \mathrm{Ad}(p_\lambda)\circ F_w$.

We will use this data to construct representations of inner forms of G. Let X_w be the preimage in $X^\vee$ of $[X^\vee/(1-w\vartheta)X^\vee]_{\mathrm{tor}}$, and let $X_{w,ad}$ be the preimage in $X_{ad}^\vee$ of $[X_{ad}^\vee/(1-w\vartheta)X_{ad}^\vee]_{\mathrm{tor}}$. We have $X^\vee \to X_{ad}^\vee \to 0$, and for $\lambda \in X_{w,ad}$, there is a unique lift $\dot{\lambda}$ of λ to X_w. Let

$$\sigma_{\dot{\lambda}} = t_{\dot{\lambda}} w\vartheta.$$

We also have $\mathcal{A} \longrightarrow \mathcal{A}_{ad} \longrightarrow 0$. Let $\tilde{x}_\lambda$ be the pre-image of x_λ in $\mathcal{A}^{\sigma_{\dot{\lambda}}}$. Let J_λ be the unique facet containing $\tilde{x}_\lambda$, as in [DR, 2.7]. Let K_λ be the parahoric subgroup of $G(K)$ determined by J_λ, and let $\mathsf{G}_\lambda = K_\lambda/K_\lambda^+$. The class $[u_\lambda] \in H^1(k, G_{ad})$ determines the inner form of G containing K_λ.

Take the element $p_\lambda \in G_{ad}(K)$ and pull it back to an element we also denote $p_\lambda \in G(K)$. We will use the notation $p_{\lambda,sc}$ if we need to distinguish the pullback from its image in $G_{ad}(K)$. Any two such choices of pullback p_λ differ by an element of the center of $G(K)$. Denote also by $\dot{w}$ an element of K_o^F, where K_o is the parahoric subgroup of $G(K)$ attached to the hyperspecial vertex o in $\mathcal{A}_{ad}$, that projects onto $\dot{w}$ as given above. Let

$$F_w = \mathrm{Ad}(\dot{w}) \circ F.$$

The parameter ϕ determines a character χ_ϕ of T^{F_w}. Define

$$T_\lambda := \mathrm{Ad}(p_\lambda)T.$$

We have $G(K)$ acts on $\mathcal{B}(G_{ad}(K))$ via the map j. By [DR, 4.4.1], $p_\lambda \cdot x_\lambda = x_\lambda$. Therefore $p_{\lambda,sc}\cdot x_\lambda = x_\lambda$ so that

$${}^0T_\lambda \subset K_\lambda$$

where ${}^0T_\lambda$ is the maximal compact subgroup of T_λ.

Conjugate the character χ_ϕ of T^{F_w} to get a character χ_λ of $T_\lambda^{F_\lambda}$: $\chi_\lambda = \chi_\phi \circ \mathrm{Ad}(p_\lambda)^{-1}$. Define

$$\pi_\lambda = c - \mathrm{Ind}_{Z^F K_\lambda^{F_\lambda}}^{G^{F_\lambda}} (\chi_\lambda \otimes \epsilon_{\mathsf{T}} \epsilon_{\mathsf{G}} R_{\mathsf{T}_\lambda, \chi_\lambda^0}^{\mathsf{G}_\lambda})$$

where $\pi_\lambda \in [\pi(\phi, \rho_\lambda)]$.

Lemma 4.1. *The representation π_λ of G^{F_λ} is irreducible supercuspidal.*

Proof. The proof is the same as [DR, 4.5.1]. □

Lemma 4.2. *The $G(K)$-orbit $[u_\lambda, \pi_\lambda] = \mathrm{Ad}(G(K)) \cdot (u_\lambda, \pi_\lambda)$ depends only on the character $\rho_\lambda \in \mathrm{Irr}(B_\phi)$.*

Proof. Let $X_{w,ad}$ denote the preimage in $X_{ad}^\vee$ of $[X_{ad}^\vee/(1 - w\vartheta)X_{ad}^\vee]_{\mathrm{tor}} \simeq \mathrm{Irr}(B_\phi)$. Given a TRD parameter ϕ and $\lambda \in X_{w,ad}$, using the DeBacker Reeder construction applied to G_{ad} choices of $C_\lambda, u_\lambda, p_\lambda, \dot{w}$ were made. In defining the representation π_λ we made choices of lifts $\dot{w}_{sc}$ and $p_{\lambda,sc}$ in $G(K)$. Given a TRD parameter ϕ, for $\lambda, \mu \in X_{w,ad}$, make choices

$$(C_\lambda, u_\lambda, p_{\lambda,sc}, \dot{w}_{\lambda,sc}), \quad (C_\mu, u_\mu, p_{\mu,sc}, \dot{w}_{\mu,sc})$$

respectively. We will show that $\rho_\lambda = \rho_\mu$ if and only if there exists a $g \in G(K)$ such that

$$g * u_\lambda = u_\mu, \quad g \cdot J_\lambda = J_\mu, \quad \mathrm{Ad}(g)_* \kappa_\lambda \simeq \kappa_\mu.$$

This is what we mean by the statement the $G(K)$-orbit $[u_\lambda, \pi_\lambda]$ depends only on ρ_λ. By [MP, 6.2] those three conditions are equivalent to having a $g \in G$ such that $\mathrm{Ad}(g) \cdot (u_\lambda, \pi_\lambda) = (u_\mu, \pi_\mu)$.

By [DR, 4.5.2] there exists a $g_{ad} \in G_{ad}(K)$ such that

$$g_{ad} * u_\lambda = u_\mu, \quad g_{ad} \cdot x_\lambda = x_\mu, \quad \mathrm{Ad}(g_{ad})(T_{ad})_\lambda = \mathrm{Ad}(s_{ad})(T_{ad})_\mu,$$

where $s_{ad} \in (T_{ad})_\mu$. As $G(K) \to G_{ad}(K)$ is surjective, choose a lift g of g_{ad} to $G(K)$. Then

$$g * u_\lambda = g_{ad} * u_\lambda = u_\mu.$$

An element $g \in G(K)$ acts on $\mathcal{B}(G_{ad}(K))$ via the adjoint quotient $j : G(K) \to G_{ad}(K)$, so

$$g \cdot x_\lambda = g_{ad} \cdot x_\lambda = x_\mu.$$

Therefore $g \cdot J_\lambda = J_\mu$. Also choose a lift s of s_{ad} to $G(K)$. Then,

$$\mathrm{Ad}(g)T_\lambda = \mathrm{Ad}(s)T_\mu, \quad \chi_\lambda \circ \mathrm{Ad}(g)^{-1} = \chi_\mu \circ \mathrm{Ad}(s)^{-1},$$

where $s \in T_\mu$. This shows $\mathrm{Ad}(g)_* \kappa_\lambda \simeq \kappa_\mu$. □

Lemma 4.3. *Given a TRD parameter ϕ for G and $\rho_\lambda \in \mathrm{Irr}(B_\phi)$, let $[u_\lambda, \pi_\lambda]$ be the associated $G(K)$-orbit of representations. For any $\rho_\lambda \in \mathrm{Irr}(A_\phi) \subset \mathrm{Irr}(B_\phi)$, let $[u_\lambda, \pi_\lambda]_{DR}$ be the $G(K)$-orbit of representations associated to ϕ and ρ_λ by DeBacker and Reeder. Then,*

$$[u_\lambda, \pi_\lambda] = [u_\lambda, \pi_\lambda]_{DR}.$$

Proof. Let ϕ be a TRD parameter for G and let $\rho_\lambda \in \mathrm{Irr}(A_\phi)$. One shows that for each step in the DeBacker and Reeder construction of a representation π_λ associated to ρ_λ, one can make concurrent choices in our construction so that

$$\pi_\lambda \in [u_\lambda, \pi_\lambda]_{DR} \iff \pi_\lambda \in [u_\lambda, \pi_\lambda].$$

□

4.5. **Construction of L_ϕ^{DR} for $GSpin_5$.** In this section let $G = GSpin_5$. Then $\hat{G} = GSp_4$, $G_{ad} = SO_5$, and $\widehat{G_{ad}} = Sp_4$. The root datum $(X, \Phi, X^\vee, \Phi^\vee)$ of $GSpin_5$ can be described as follows [AS, Prop 2.1]. We have that

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^*,$$

$$\Delta = \{a_1 = e_1 - e_2, a_2 = e_2\}, \quad \Delta^\vee = \{a_1^\vee = e_1^* - e_2^*, a_2^\vee = 2e_2^* - e_0^*\}$$

are the character and cocharacter lattices, and simple roots and coroots, respectively. Then

$$T(K) = \{\prod_{j=0}^{2} e_j^*(\lambda_j) : \lambda_j \in K^\times\}$$

is the set of K points of a maximal K-split torus T of G.

Let $T_{ad} = T/Z$ where Z is the center of G. Then $X^*(T_{ad}) \hookrightarrow X^*(T)$ is the submodule $\bigoplus a_i e_i$ such that $a_0 = 0$. This submodule contains the simple roots $e_1 - e_2$ and e_2. Also,

$$X_*(T) \to X_*(T_{ad}) = \big(\bigoplus_i \mathbb{Z}e_i^*\big)/\mathbb{Z}e_0^*.$$

The simple coroots for T_{ad} are $e_1^* - e_2^*$ and $2e_2^*$, the image of the coroots for T. We see that

$$X_{ad} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2, \quad X_{ad}^\vee = \mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^*, \quad \Delta = \{e_1 - e_2, e_2\}, \quad \Delta^\vee = \{e_1^* - e_2^*, 2e_2^*\}$$

gives a root datum for SO_5. We have

$$\mathcal{A} = \mathbb{R}e_0^* \oplus \mathbb{R}e_1^* \oplus \mathbb{R}e_2^*, \quad \mathcal{A}_{ad} = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^*,$$

where the projection $j : \mathcal{A} \to \mathcal{A}_{ad}$ is given by $(x_0, x_1, x_2) \mapsto (x_1, x_2)$. The fundamental chamber C is given by the inequalities

$$1 - e_2 > e_1 > e_2 > 0.$$

We have the short exact sequence

$$1 \to GL_1 \to GSpin_5 \to SO_5 \to 1$$

which gives

$$1 \to GL_1(K) \to GSpin_5(K) \to SO_5(K) \to \mathrm{H}^1(K, GL_1) = 0$$

so that

$$GSpin_5(K) \to SO_5(K) \to 1$$

is surjective. We apply the construction of the previous section.

Let ϕ be a TRD parameter for $GSpin_5$. If ϕ is irreducible, $\hat{w}$ is a Coxeter element. Precisely, if s_1 is the simple reflection corresponding to the simple root $e_1 - e_2$ for GSp_4 and s_2 is the simple reflection corresponding to the simple root $2e_2 - e_0$ for GSp_4, then $\hat{w} = s_1 s_2$ or $s_2 s_1$. If $\phi = \phi_1 \oplus \phi_2$ then $\hat{w} = s_1 s_2 s_1 s_2$. If ϕ is irreducible, or $\phi = \phi_1 \oplus \phi_2$ respectively, as automorphisms of X,

$$\hat{w}e_0 = e_0 + e_2, \quad \hat{w}e_1 = -e_2, \quad \hat{w}e_2 = e_1,$$

$$\hat{w}e_0 = e_0 + e_1 + e_2, \quad \hat{w}e_1 = -e_1, \quad \hat{w}e_2 = -e_2.$$

The automorphism $w \in \mathrm{Aut}(X_{ad}^\vee)$ is defined by the equations $\langle e_i, we_j^* \rangle = \langle \hat{w}e_i, e_j^* \rangle$ where $i, j = 1, 2$. Then,

$$we_1^* = e_2^*, \quad we_2^* = -e_1^*, \quad \phi \text{ irreducible}$$

$$we_1^* = -e_1^*, \quad we_2^* = -e_2^*, \quad \phi = \phi_1 \oplus \phi_2.$$

One can compute

$$[X_{ad}^\vee/(1-w)X_{ad}^\vee]_{\rm tor} = \{\overline{0}, \overline{e_1^*} = \overline{e_2^*}\} \cong \mathrm{Irr}(B_\phi), \quad \phi \text{ irreducible}$$
$$[X_{ad}^\vee/(1-w)X_{ad}^\vee]_{\rm tor} = \{\overline{0}, \overline{e_1^*+e_2^*}, \overline{e_1^*}, \overline{e_2^*}\} \cong \mathrm{Irr}(B_\phi), \quad \phi = \phi_1 \oplus \phi_2.$$

Let $\rho_\lambda \in [X_{ad}^\vee/(1-w)X_{ad}^\vee]_{\rm tor}$. We obtain an element $x_\lambda \in \mathcal{A}_{ad}$ and a facet $J_\lambda \in \mathcal{A}$. If K_λ is the parahoric subgroup of $GSpin_5(K)$ determined by J_λ, we can determine the root datum of the group $\mathsf{G}_\lambda := K_\lambda/K_\lambda^+$ as follows. Let J be a facet in the fundamental chamber C. The reduction mod $\mathfrak{p}$ of T, denoted T, is a maximal $\bar{\mathfrak{f}}$-split torus of the reduction mod $\mathfrak{p}$ of G^J. The character group of T is canonically isomorphic with the character group of T. The positive roots of G_J are

$$\Phi_J^+ = \{a \in \Phi^+ : \langle a, x\rangle \in \mathbb{Z} \text{ for all } x \in J\},$$

where Φ^+ is a set of positive roots for G given by a choice of simple roots Δ. The coroot associated with a root $a \in \Phi_J^+$ is the same for G_J as for G. See the appendix for the determination of root datum for specific connected reductive linear algebraic groups that occur in this paper.

The affine Weyl group W_{ad} decomposes as $W_{ad} = \Omega_C W^\circ$, where W° acts simply transitively on the alcoves in $\mathcal{A}_{ad}$ and $\Omega_C = \{\omega \in W : \omega\cdot C = C\}$. Up to conjugation to an automorphism of the fundamental chamber $C \subset \mathcal{A}_{ad}$ the element $y_\lambda \in \Omega_C$. For $G_{ad} = SO_5$, we have $\Omega_C = \{1, -1\}$ where -1 acts on $\mathcal{A}_{ad}$ as reflection in the first coordinate

$$-1\cdot(x_1, x_2) = (1-x_1, x_2).$$

The class $[u_\lambda] \in H^1(k, G_{ad})$, where $u_\lambda \in N_{G_{ad}(K)}(T_{ad}(K))$ is a lift of y_λ, determines the inner form of $GSpin_5$ containing K_λ. We have the following:

ϕ	ρ_λ	x_λ	$\Phi_{x_\lambda}^+$	$[u_\lambda]$
irred	$\overline{0}$	0	$\{e_1-e_2, e_2, e_1, e_1+e_2\}$	1
irred	$\overline{e_1^*}$	$1/2(e_1^*+e_2^*)$	$\{e_1-e_2, e_1+e_2\}$	-1
$\phi_1\oplus\phi_2$	$\bar{0}$	0	$\{e_1-e_2, e_2, e_1, e_1+e_2\}$	1
$\phi_1\oplus\phi_2$	$\overline{e_1^*+e_2^*}$	$1/2(e_1^*+e_2^*)$	$\{e_1-e_2, e_1+e_2\}$	1
$\phi_1\oplus\phi_2$	$\overline{e_1^*}$	$1/2(e_1^*)$	$\{e_2\}$	-1
$\phi_1\oplus\phi_2$	$\overline{e_2^*}$	$1/2(e_2^*)$	$\{e_1\}$	-1

In the following table we view $\rho_\lambda \in \mathrm{Irr}(B_\phi)$. Note that

$${}^2GSpin_4 \cong GL_2(\mathbb{F}_{q^2})^\circ = \{g \in GL_2(\mathbb{F}_{q^2}) : \det(g) \in \mathbb{F}_q^\times\}, \quad {}^2GSpin_2 \cong \mathbb{F}_{q^2}^\times.$$

We have:

ϕ	ρ_λ	$\mathsf{G}_\lambda^{F_\lambda}$	G^{F_λ}
irred	(1)	$GSpin_5(\mathfrak{f})$	$GSpin_5(k)$
irred	(-1)	${}^2GSpin_4(\mathfrak{f})$	$GSpin_{4,1}(k)$
$\phi_1 \oplus \phi_2$	$(1,1)$	$GSpin_5(\mathfrak{f})$	$GSpin_5(k)$
$\phi_1 \oplus \phi_2$	$(-1,-1)$	$GSpin_4(\mathfrak{f})$	$GSpin_5(k)$
$\phi_1 \oplus \phi_2$	$(-1,1)$	$[({}^2GSpin_2 \times GSpin_3)/\Delta GL_1](\mathfrak{f})$	$GSpin_{4,1}(k)$
$\phi_1 \oplus \phi_2$	$(1,-1)$	$[({}^2GSpin_2 \times GSpin_3)/\Delta GL_1](\mathfrak{f})$	$GSpin_{4,1}(k)$

The following table lists the data for the irreducible cuspidal Deligne-Lusztig representation $\epsilon_{\mathsf{T}}\epsilon_{\mathsf{G}}R_{\mathsf{T}_\lambda,\chi_\lambda^0}^{\mathsf{G}_\lambda}$ of $\mathsf{G}_\lambda^{F_\lambda}$. The notation $+,-,\dagger,\ddagger$ is introduced in Section 5.3 and will be used in the remainder of the paper to denote the various cases. The character χ_λ of $\mathsf{T}_\lambda^{F_\lambda}$ is given by an element t in the F_λ^*-stable dual torus $\mathsf{T}_\lambda^{*F_\lambda^*}$. We give the element s in the F_w^*-stable torus $\mathsf{T}^{*F_w^*}$ such that t is conjugate to s. We have

$$\tau \in \mathbb{F}_{q^4}^\times \setminus \mathbb{F}_{q^2}^\times,$$

and

$$\tau_1, \tau_2 \in \mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q^\times, \quad N_{\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times}(\tau_1) = N_{\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times}(\tau_2) \quad \text{and} \quad \tau_1 \neq \tau_2, \tau_2^q.$$

ϕ	$\pm R_{\mathsf{T}_\lambda^*}(t)$	s	$\mathsf{G}_\lambda^{F_\lambda}$
irred	R_π^+	$\mathrm{diag}(\tau,\tau^q,\tau^{q^3},\tau^{q^2})$	$GSpin_5(\mathfrak{f})$
irred	$R_\pi^\dagger$	$\mathrm{diag}(\tau,\tau^q,\tau^{q^3},\tau^{q^2})$	${}^2GSpin_4(\mathfrak{f})$
$\phi_1 \oplus \phi_2$	R_π^+	$\mathrm{diag}(\tau_1,\tau_2,\tau_2^q,\tau_1^q)$	$GSpin_5(\mathfrak{f})$
$\phi_1 \oplus \phi_2$	R_π^-	$\mathrm{diag}(\tau_1,\tau_2,\tau_2^q,\tau_1^q)$	$GSpin_4(\mathfrak{f})$
$\phi_1 \oplus \phi_2$	$R_\pi^\ddagger$	$\mathrm{diag}(\tau_1,\tau_2,\tau_2^q,\tau_1^q)$	$[({}^2GSpin_2 \times GSpin_3)/\Delta GL_1](\mathfrak{f})$
$\phi_1 \oplus \phi_2$	$R_\pi^\ddagger$	$\mathrm{diag}(\tau_2,\tau_1,\tau_1^q,\tau_2^q)$	$[({}^2GSpin_2 \times GSpin_3)/\Delta GL_1](\mathfrak{f})$

For each $\rho_\lambda \in \mathrm{Irr}(B_\phi)$ we have

$$\pi_\lambda = c - \mathrm{Ind}_{Z^F K_\lambda^{F_\lambda}}^{G^{F_\lambda}}\left(\chi_\lambda \otimes \epsilon_{\mathsf{T}}\epsilon_{\mathsf{G}}R_{\mathsf{T}_\lambda^*}^{\mathsf{G}_\lambda}(t)\right)$$

where $\pi_\lambda \in [\pi(\phi,\rho_\lambda)]$. Recall $\Pi(\phi,\omega) = \{\pi(\phi,\rho) : \rho \in \mathrm{Irr}(B_\phi), \omega_\rho = \omega\}$. Let

$$L_\phi^{DR} = \Pi(\phi,1) \quad \text{or} \quad \Pi(\phi,-1)$$

be the L-packet of depth zero supercuspidal representations of $GSpin_5$ or $GSpin_{4,1}$, respectively, attached to the TRD parameter ϕ by the DeBacker-Reeder construction.

5. The generalized principal series $I(s, \pi \boxtimes \sigma)$

5.1. **Tame regular discrete series local Langlands for GL_{2m}.** Let $\phi = \phi_1 \oplus \cdots \oplus \phi_r$, where $r = 1, 2$, be a tame regular discrete series Langlands parameter for $GSpin_5$. Then, for $1 \leq i \leq r$,

$$\phi_i : W_k \longrightarrow GSp_{2m}(\mathbb{C}) \hookrightarrow GL_{2m}(\mathbb{C}) = GL_{2m}(k)^\vee$$

is also Langlands parameter for $GL_{2m}(k)$. By the local Langlands correspondence for GL_{2m}, let

$$\phi_i \longleftrightarrow L_{\phi_i} = \{\sigma_{\phi_i}\},$$

where the L-packets for GL_{2m} are always of size one.

By Section 3, $\phi_i = \mathrm{Ind}_{W_{k_{2m}}}^{W_k} \eta$, where η is a regular character of $W_{k_{2m}}$. By [He1], the representation σ_{ϕ_i} is the irreducible depth zero supercuspidal representation $\pi(\eta) \otimes \omega$, where $\pi(\eta)$ is the representation attached to ϕ_i by Gerardin [Ge], and ω is the unramified character of order two of the extension k_{2m} over k.

Under the Artin map, the character η of $W_{k_{2m}}$ corresponds to a character, which we will also denote η, of $k_{2m}^\times$. As η is tame, $\eta|_{\mathfrak{o}_{k_{2m}}^\times}$ factors through a character of $\mathfrak{f}_{2m}^\times$. We can view $\mathfrak{f}_{2m}^\times \subset GL_{2m}(\mathfrak{f})$ as the group of $\mathfrak{f}$-points of a minisotropic torus S of $GL_{2m}(\bar{\mathfrak{f}})$ which is defined over $\mathfrak{f}$. As η is regular, as a character of $\mathfrak{f}_{2m}^\times$ it is in general position, and the Deligne-Lusztig character $\epsilon_{\mathsf{G}}\epsilon_{\mathsf{S}} R_{\mathfrak{f}_{2m}^\times, \eta}$ gives an irreducible cuspidal representation of $GL_{2m}(\mathfrak{f})$. Then

$$\pi(\eta) = c - \mathrm{Ind}_{k^\times GL_{2m}(\mathfrak{o})}^{GL_{2m}(k)} \eta \otimes \epsilon_{\mathsf{G}}\epsilon_{\mathsf{S}} R_{\mathfrak{f}_{2m}^\times, \eta}.$$

Let $\mathsf{T} \subset GL_{2m}(\bar{\mathfrak{f}})$ be the split maximal torus. We can view the character η as a regular element t in the dual F^* stable torus S^{*F^*}, where t is conjugate over $GL_{2m}(\bar{\mathfrak{f}})$ to an element s in the Coxeter torus $\mathsf{T}^{*F_w^*}$. Here the regular Frobenius action F is twisted by the Coxeter element of the Weyl group, and we have the dual action

$$F_w^*(x_1, x_2, \ldots, x_{2m}) = (x_{2m}^q, x_1^q, \ldots, x_{2m-1}^q).$$

Given a tame regular discrete series parameter ϕ, we list the corresponding element s, where τ, τ_1, and τ_2 are as in the previous section.

ϕ_i	$\pm R_{\mathsf{S}^*}(t)$	s	$GL_{2m}(\mathfrak{f})$
4 dim	$R_\sigma^{+\mathrm{or}\dagger}$	$\mathrm{diag}(\tau, \tau^q, \tau^{q^2}, \tau^{q^3})$	$GL_4(\mathfrak{f})$
2 dim	$R_\sigma^{+,-,\mathrm{or}\ddagger}$	$\mathrm{diag}(\tau_1, \tau_1^q)$	$GL_2(\mathfrak{f})$
2 dim	$R_\sigma^{+,-,\mathrm{or}\ddagger}$	$\mathrm{diag}(\tau_2, \tau_2^q)$	$GL_2(\mathfrak{f})$

5.2. **The Bernstein component of $I(s, \pi \boxtimes \sigma)$.** From now on, let $G = GSpin_{4m+5}(k)$ or $GSpin_{2m+4,2m+1}(k)$. In the following, we will always assume $m = 1$ or 2. For simplicity of notation, let $n = 2m + 2$. Let $P = M \cdot N$ be the maximal parabolic subgroup of G with Levi factor

$$M = GSpin_5(k) \times GL_{2m}(k) \quad \text{if} \quad G = GSpin_{4m+5}(k);$$

$$M = GSpin_{4,1}(k) \times GL_{2m}(k) \quad \text{if} \quad G = GSpin_{2m+4,2m+1}(k).$$

Note that in each case P corresponds to the simple root $a_{2m} = e_{2m} - e_{2m+1}$. If π is an irreducible representation of $GSpin_5(k)$ or $GSpin_{4,1}(k)$, and σ a representation of $GL_{2m}(k)$ we can form the generalized principal series representation

$$I(s, \pi \boxtimes \sigma) = \mathrm{Ind}_P^G \delta_P^{1/2} \pi \boxtimes \sigma |\det|^s,$$

where $s \in \mathbb{C}$.

Let $\mathscr{B}(G)$ be the set of classes of irreducible supercuspidal representations of rational Levi components of rational parabolic subgroups of G under the equivalence given from G-conjugation and twisting by unramified quasicharacters of the Levi components. The inertial support of an irreducible representation of G is the inertial equivalence class $\mathfrak{s} \in \mathscr{B}(G)$ of the support of the representation. The theory of the Bernstein centre [Be] decomposes $\mathfrak{R}(G)$, the category of smooth complex representations of G, into subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$, where the objects of $\mathfrak{R}^{\mathfrak{s}}(G)$ are the smooth representations of G all of whose irreducible subquotients have inertial support $\mathfrak{s}$. Any unramified quasicharacter χ of $GSpin_5 \times GL_{2m}$ or $GSpin_{4,1} \times GL_{2m}$ is of the form

$$\chi = |\mathrm{sim}|^t \boxtimes |\det|^s, \quad s, t \in \mathbb{C},$$

where sim and det are k-rational characters of $GSpin_5$ or $GSpin_{4,1}$ and GL_{2m}, respectively. Therefore, the Bernstein component of $I(s, \pi \boxtimes \sigma)$ is the set of all smooth representationss κ of G such that all the irreducible subquotients of κ are a composition factor of a representation equivalent to

$$\mathrm{Ind}_P^G \delta_P^{1/2} \pi |\mathrm{sim}|^t \boxtimes \sigma |\det|^s,$$

for some $s, t \in \mathbb{C}$.

From now on let $I(s, \pi \boxtimes \sigma)$ be the generalized principal series where given a tame regular discrete series parameter ϕ, $\pi \in L_\phi^{DR}$ where L_ϕ^{DR} is the L-packet of representations of $GSpin_5(k)$ or $GSpin_{4,1}(k)$ as given in Section 4.5, and $\sigma = \sigma_{\phi_i}$ as given in Section 5.1.

In [Mo1], Morris shows that if $\mathcal{P}$ is a parahoric subgroup of a connected reductive k-group G, where the reductive quotient of $\mathcal{P}$ is denoted M, and ρ is an irreducible cuspidal reprentation of M, then $(\mathcal{P}, \rho)$ is a $\mathfrak{S}$-type for a finite set of inertial equivalence classes $\mathfrak{S} \subset \mathscr{B}(G)$. Using this work of Morris, we give a parahoric subgroup $\mathcal{P}$ of G and an irreducible cuspidal representation ρ of the reductive quotient M, such that $(\mathcal{P}, \rho)$ is a $[M, \pi \boxtimes \sigma]_G$-type in G. Then, using the theory of types and covers developed by Bushnell and Kutzko [BK2], we have that representations in the Bernstein component of $I(s, \pi \boxtimes \sigma)$ are parametrized by unital left modules of $\mathcal{H}(G, \rho)$ the Hecke algebra of compactly supported ρ-spherical functions on G. Namely,

$$\mathcal{H}(G, \rho) = \{f \in \mathcal{C}_c^\infty(G, \mathrm{End}_{\mathbb{C}}(W^\vee)) |\, f(k_1 g k_2) = \check{\rho}(k_1) f(g) \check{\rho}(k_2), \quad k_i \in \mathcal{P},\, g \in G\},$$

where $\mathcal{C}_c^\infty(G, \mathrm{End}_{\mathbb{C}}(W^\vee))$ is the set of functions $f : G \to \mathrm{End}_{\mathbb{C}}(W^\vee)$ such that f is locally constant with compact support and $(\check{\rho}, W^\vee)$ denotes the contragredient representation. As algebras,

$$\mathcal{H}(G, \rho) \cong \mathrm{End}_G(c - \mathrm{Ind}_{\mathcal{P}}^G(\rho)).$$

5.3. **Definition of $\mathcal{P}$.** The vertices of the local Dynkin diagram for $G = GSpin_{4m+5}$ are in correspondence with the set

$$\Pi^+ = \{\alpha_{0^+} = -e_1 - e_2 + 1, \alpha_{1^+} = e_1 - e_2, \alpha_{2^+} = e_2 - e_3, \ldots, \alpha_{n-1^+} = e_{n-1} - e_n, \alpha_{n^+} = e_n\}.$$

By Bruhat-Tits theory, let $\mathcal{K}_{i^+}$ be the parahoric subgroup of $GSpin_{4m+5}(k)$ corresponding to the root α_{i^+} in the local Dynkin diagram. Let

$$\mathcal{P}^+ = \mathcal{K}_{0^+} \cap \mathcal{K}_{2m^+}.$$

Another choice of simple roots for $GSpin_9$ is

$$\Delta^- = \{e_3 - e_4, e_4 - e_1, e_1 - e_2, e_2\}.$$

Let $\alpha_{0^-} = -e_3 - e_4 + 1$ where $-e_3 - e_4$ is the lowest root. Let $\mathcal{K}_{i^-}$ be the parahoric subgroup of $GSpin_9(k)$ corresponding to the root α_{i^-} in the local Dynkin diagram corresponding to the set

$$\Pi^- = \{\alpha_{0^-} = -e_3 - e_4 + 1, \alpha_{1^-} = e_3 - e_4, \alpha_{2^-} = e_4 - e_1, \alpha_{3^-} = e_1 - e_2, \alpha_{4^-} = e_2\}.$$

Let

$$\mathcal{P}^- = \mathcal{K}_{2^-} \cap \mathcal{K}_{4^-}.$$

The vertices of the relative local Dynkin diagram for $G = GSpin_{2m+4,2m+1}$ are in correspondence with the set

$$\{\alpha_0 = 1 - e_1, \alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{2m} = e_{2m} - e_{2m+1}, \alpha_{2m+1} = e_{2m+1}\}.$$

Note that this is a set of simple affine roots for an affine Weyl group of type C_{2m+1}. Another choice of simple roots for S, where S is the maximal k-split torus of $GSpin_{8,5}$ gives us another set of simple affine roots

$$\Pi^\dagger = \{\alpha_{0^\dagger} = 1 - e_5, \alpha_{1^\dagger} = e_5 - e_1, \alpha_{2^\dagger} = e_1 - e_2, \alpha_{3^\dagger} = e_2 - e_3, \alpha_{4^\dagger} = e_3 - e_4, \alpha_{5^\dagger} = e_4\}.$$

Let

$$\mathcal{P}^\dagger = \mathcal{K}_{1^\dagger} \cap \mathcal{K}_{5^\dagger}.$$

The vertices of the local Dynkin diagram for $GSpin_{6,3}$ are in correspondence with the set

$$\Pi^\ddagger = \{\alpha_{0^\ddagger} = 1 - e_1, \alpha_{1^\ddagger} = e_1 - e_2, \alpha_{2^\ddagger} = e_2 - e_3, \alpha_{3^\ddagger} = e_3\}.$$

Let

$$\mathcal{P}^\ddagger = \mathcal{K}_{0^\ddagger} \cap \mathcal{K}_{2^\ddagger}.$$

Denote $\mathcal{P} = \mathcal{P}^a, a = +, -, \dagger, \ddagger$, depending on the context, so

$$\mathcal{Q} = \mathcal{P} \cap M = K_\lambda^{F_\lambda} \times GL_{2m}(\mathfrak{o}).$$

Let $\mathsf{M} = \mathsf{G}_\lambda^{F_\lambda} \times GL_{2m}(\mathfrak{f})$ be the reductive component of the reduction mod $\mathfrak{p}$ of $\mathcal{P}$ which is equal to the reductive component of the reduction mod $\mathfrak{p}$ of $\mathcal{Q}$. If $\mathcal{P} = \mathcal{P}^a$, denote by

$$\rho = R_\pi \boxtimes R_\sigma = R_\pi^a \boxtimes R_\sigma^a, \quad a = +, -, \dagger, \ddagger,$$

a representation of M, where the R^a are given in the tables in Section 4.5 and Section 5.1. We can view ρ as a representation of $\mathcal{P}$ via inflation. We can also view ρ as a representation of $\mathcal{Q}$ via inflation. We denote this representation of $\mathcal{Q}$ by ρ_M.

Lemma 5.1. *The pair $(\mathcal{Q}, \rho_M)$ is a type for the inertial class $[M, \pi \boxtimes \sigma]_M$ in M and $(\mathcal{P}, \rho)$ is a G-cover for it. Therefore, the pair $(\mathcal{P}, \rho)$ is a type for the inertial class $[M, \pi \boxtimes \sigma]_G$ in G.*

Proof. Since $((\chi_\lambda \otimes R_\pi) \boxtimes (\eta \otimes R_\sigma))|_{\mathcal{Q}} = \rho_M$ is irreducible, by [BK2, Prop 5.4] $(\mathcal{Q}, \rho_M)$ is a type for the inertial class $[M, \pi \boxtimes \sigma]_M$ in M. It is shown in [Mo1, pg. 149] that $(\mathcal{P}, \rho)$ is a G-cover for $(\mathcal{Q}, \rho_M)$. Then, by [BK2, Thm 8.3], $(\mathcal{P}, \rho)$ is a type for the inertial class $[M, \pi \boxtimes \sigma]_G$ in G. $\square$

Corollary 5.2. *The Hecke algebra of the Bernstein component of $I(s, \pi \boxtimes \sigma)$ is $\mathcal{H}(G, \rho)$.*

Proof. We have
$$\mathfrak{R}^{[M,\pi\boxtimes\sigma]_G}(G) = \mathfrak{R}_\rho(G) \cong \mathcal{H}(G,\rho)\text{−Mod},$$
where $[M, \pi \boxtimes \sigma]_G$ is the inertial support of $I(s, \pi \boxtimes \sigma)$. The last equivalence is given by the functor M_ρ restricted to $\mathfrak{R}_\rho(G)$. □

6. The Hecke algebra $\mathcal{H}(G, \rho)$

6.1. **A presentation of $\mathcal{H}(G, \rho)$.** In [Mo2], Morris describes explicit generators and relations for $\mathcal{H}(G, \rho)$ when ρ is an irreducible cuspidal representation of the reductive quotient of a parahoric subgroup $\mathcal{P}$ of G. Let Π be a set of simple affine roots in correspondence with the vertices of the relative local Dynkin diagram for G. For $\Theta \subset \Pi$ define $W_\Theta = \langle s_\alpha \,|\, \alpha \in \Theta\rangle$. Define
$$S_\Theta = \{w \in N_W(W_\Theta) \,|\, w\Theta = \Theta\},$$
where $N_W(W_\Theta)$ is the normalizer of W_Θ in the affine Weyl group W. Let $\mathcal{P}$ corresponds to $\Theta \subset \Pi$, and let M be the reductive quotient of $\mathcal{P}$. By [Mo2, §4] $\mathcal{H}(G, \rho)$ is supported on double cosets $\mathcal{P}\dot{w}\mathcal{P}$ where $\dot{w} \in N_G(T)$ such that under the induced action on M, ${}^{\dot{w}}\mathsf{M} = \mathsf{M}$ as representations of $\mathsf{M}, \dot{w}\rho \cong \rho$, and $\dot{w}$ projects to an element $w \in S_\Theta$.

Suppose that $J \subset \Pi$ such that $wJ = \Theta$ for some $w \in W$, and $\alpha \in \Pi$. Let t be the longest element in the Weyl group W_J corresponding to the spherical root system obtained from J such that $t^2 = 1$ and $t(J) = -J$. Let u be the longest element in the Weyl group $W_{J\cup\{\alpha\}}$ corresponding to the spherical root system obtained from $J \cup \{\alpha\}$ such that $u^2 = 1$ and $u(J \cup \{\alpha\}) = -(J \cup \{\alpha\})$. Set
$$v[\alpha, J] = u \cdot t \in W_{J\cup\alpha} \subset W.$$

We now specialize to the case of interest. Denote by $\Theta = \Theta^+ \subset \Pi^+, \Theta^- \subset \Pi^-, \Theta^\dagger \subset \Pi^\dagger, \Theta^\ddagger \subset \Pi^\ddagger$:
$$\Theta^+ = \{\alpha_{0^+}, \dots, \alpha_{2m+2^+}\} \setminus \{\alpha_{0^+}, \alpha_{2m^+}\}, \quad \Theta^- = \{\alpha_{0^-}, \dots, \alpha_{4^-}\} \setminus \{\alpha_{2^-}, \alpha_{4^-}\},$$
$$\Theta^\dagger = \{\alpha_{0^\dagger}, \dots, \alpha_{5^\dagger}\} \setminus \{\alpha_{1^\dagger}, \alpha_{5^\dagger}\}, \quad \Theta^\ddagger = \{\alpha_{0^\ddagger}, \dots \alpha_{3^\ddagger}\} \setminus \{\alpha_{0^\ddagger}, \alpha_{2^\ddagger}\}.$$

Lemma 6.1.
- (i) $S_{\Theta^+} = \langle v[\alpha_{0^+}, \Theta^+], v[\alpha_{2m^+}, \Theta^+], T(e_0^*)\rangle$;
- (ii) $S_{\Theta^-} = \langle v[\alpha_{2^-}, \Theta^-], v[\alpha_{4^-}, \Theta^-], \nu\rangle$ *where* $\nu = T(e_3^*)s_{\alpha_{1^-}} s_{\alpha_{2^-}} s_{\alpha_{3^-}} s_{\alpha_{4^-}} s_{\alpha_{3^-}} s_{\alpha_{2^-}} s_{\alpha_{1^-}}$, *is a diagram automorphism preserving the fundamental chamber corresponding to* Π^- *such that* $\nu^2 = T(e_0^*)$ *is translation in the central direction;*
- (iii) $S_{\Theta^\dagger} = \langle v[\alpha_{1^\dagger}, \Theta^\dagger], v[\alpha_{5^\dagger}, \Theta^\dagger], T(e_0^*)\rangle$;
- (iv) $S_{\Theta^\ddagger} = \langle v[\alpha_{0^\ddagger}, \Theta^\ddagger], v[\alpha_{2^\ddagger}, \Theta^\ddagger], T(e_0^*)\rangle$.

In all cases the elements $v[\alpha_i, \Theta]$ are involutions, and $T(e_0^*)$ is translation by e_0^*.

Proof. For $J \subset \Pi$, corresponding to a connected piece of the extended Dynkin diagram of type $B_n, C_n, D_n (n \text{ even})$, $u(\alpha_j) = -\alpha_j$ for $\alpha_j \in J$. Here, u is the longest element in the Weyl group W_J defined above. As the piece of the extended Dynkin diagram corresponding to $\Theta^+ \cup \{\alpha_{2m^+}\}$ is of type B_{2m+2},
$$v[\alpha_{2m^+}, \Theta^+]\Theta^+ = \Theta^+.$$

By [Ho, Lem 10], $v[\alpha_{0^+},\Theta^+] = v[\alpha_{0^+},\{\alpha_{1^+},\ldots,\alpha_{2m-1^+}\}]$, therefore for $b\in\{\alpha_{2m+1^+},\alpha_{2m+2^+}\}$, $v[\alpha_{0^+},\Theta^+](b)=b$, and for $b\in\{\alpha_{1^+},\ldots,\alpha_{2m-1^+}\}, v[\alpha_{0^+},\Theta^+](b)=b$, so

$$v[\alpha_{0^+},\Theta^+]\Theta^+ = \Theta^+$$

($\{\alpha_{1^+},\ldots,\alpha_{2m-1^+}\}\cup\{\alpha_{0^+}\}$ corresponds to a diagram of type D_{2m}). Since $v[\alpha_i,\Theta^+]\Theta^+=\Theta^+$ for $i=0^+,2m^+$, By [Mo2, Lem 2.4(c)] they are involutions.

Now, by [Mo2, Lem 2.5], if $w\in W$ such that $w\Theta=\Theta$, we can find $J_1,\ldots,J_{r+2}$, where $J_i\subset\Pi$ and $\Theta=J_1=J_{r+2}$, and $\alpha_1,\ldots\alpha_r\in\Pi$ such that $v[\alpha_i,J_i]J_i=J_{i+1}, 1\le i\le r$, and $w=\nu v[\alpha_r,J_r]\ldots v[\alpha_1,J_1]$ where $\nu\in\Omega, \nu J_{r+1}=J_{r+2}=\Theta$. Here, $\nu\in\Omega=\{w\in W\,|\,wC=C\}$ the set of diagram automorphisms that fix the fundamental chamber C. For $GSpin_{4m+5}$, for the simple roots given by Δ^+, the set $\Omega=\langle\nu\rangle$, where

$$\nu = T(e_1^*)s_{\alpha_{1^+}}\ldots s_{\alpha_{n-1^+}}s_{\alpha_{n^+}}s_{\alpha_{n-1^+}}\ldots s_{\alpha_{1^+}}$$

is a diagram automorphism preserving the fundamental chamber given by Δ^+. We have $\nu^2=T(e_0^*)$ is translation in the central direction. The action of ν on the simple affine roots is as follows,

$$\nu\cdot\alpha_{0^+}=\alpha_{1^+}, \nu\cdot\alpha_{1^+}=\alpha_{0^+}, \nu\cdot\alpha_{i^+}=\alpha_{i^+},\ i>1.$$

Therefore, ν does not preserve Θ^+. However, $\nu^2=T(e_0^*)$ does preserve Θ^+. Since $v[\alpha_i,\Theta^+]\Theta^+=\Theta^+$ for all $\alpha_i\in\Pi^+\setminus\Theta^+$, $J_1=J_2=\cdots=J_{r+1}=\Theta^+$. Therefore, if $w\Theta^+=\Theta^+$, w is a word in $v[\alpha_{0^+},\Theta^+], v[\alpha_{2m^+},\Theta^+]$, and $T(e_0^*)$. This shows (i).

For cases (ii), (iii), (iv), the proof is as in (i). As the piece of the extended Dynkin diagram corresponding to $\Theta^-\cup\{\alpha_{2^-}\}$ is of type D_4, $v[\alpha_{2^-},\Theta^-]\Theta^-=\Theta^-$. By [Ho, Lem 10], $v[\alpha_{4^-},\Theta^-]=v[\alpha_{4^-},\{\alpha_{3^-}\}]$ and $v[\alpha_{4^-},\Theta^-]\Theta^-=\Theta^-$ ($\{\alpha_{3^-}\}\cup\{\alpha_{4^-}\}$ corresponds to a diagram of type B_2). As the piece of the extended Dynkin diagram corresponding to $\Theta^\dagger\cup\{\alpha_{1^\dagger}\}$ is of type B_5, $v[\alpha_{1^\dagger},\Theta^\dagger]\Theta^\dagger=\Theta^\dagger$. By [Ho, Lem 10], $v[\alpha_{5^\dagger},\Theta^\dagger]=v[\alpha_{5^\dagger},\{\alpha_{2^\dagger},\alpha_{3^\dagger},\alpha_{4^\dagger}\}]$ and $v[\alpha_{5^\dagger},\Theta^\dagger]\Theta^\dagger=\Theta^\dagger$ ($\{\alpha_{2^\dagger},\alpha_{3^\dagger},\alpha_{4^\dagger}\}\cup\{\alpha_{5^\dagger}\}$ corresponds to a diagram of type B_4). As the piece of the extended Dynkin diagram corresponding to $\Theta^\ddagger\cup\{\alpha_{2^\ddagger}\}$ is of type B_3, $v[\alpha_{2^\ddagger},\Theta^\ddagger]\Theta^\ddagger=\Theta^\ddagger$. By [Ho, Lem 10], $v[\alpha_{0^\ddagger},\Theta^\ddagger]=v[\alpha_{0^\ddagger},\{\alpha_{1^\ddagger}\}]$ and $v[\alpha_{0^\ddagger},\Theta^\ddagger]\Theta^\ddagger=\Theta^\ddagger$ ($\{\alpha_{1^\ddagger}\}\cup\{\alpha_{0^\ddagger}\}$ corresponds to a diagram of type B_2).

When the simple roots are given by Δ^-, the set of diagram automorphisms $\Omega=\langle\nu\rangle$, where

$$\nu = T(e_3^*)s_{\alpha_{1^-}}s_{\alpha_{2^-}}s_{\alpha_{3^-}}s_{\alpha_{4^-}}s_{\alpha_{3^-}}s_{\alpha_{2^-}}s_{\alpha_{1^-}}$$

is a diagram automorphism preserving the fundamental chamber given by Δ^- such that $\nu^2=T(e_0^*)$ is translation in the central direction. The action of ν on the simple roots is as follows,

$$\nu\cdot\alpha_{0^-}=\alpha_{1^-}, \nu\cdot\alpha_{1^-}=\alpha_{0^-}, \nu\cdot\alpha_{i^-}=\alpha_{i^-}, i>1.$$

Therefore in case (ii), ν does preserve Θ^-. In cases (iii) and (iv), $\Omega=\langle T(e_0^*)\rangle$ and $T(e_0^*)$ preserves both $\Theta^\dagger$ and $\Theta^\ddagger$. ☐

Lemma 6.2. (i) *For $i=0^+,2m^+$ if $\Theta=\Theta^+$, $i=2^-,4^-$ if $\Theta=\Theta^-$, $i=1^\dagger,5^\dagger$ if $\Theta=\Theta^\dagger$, and $i=0^\ddagger,2^\ddagger$ if $\Theta=\Theta^\ddagger$, we have*

$$v[\alpha_i,\Theta]\cdot\rho\cong\rho,$$

for $\rho=R_\pi\boxtimes R_\sigma$ a representation of M as given in Section 5.3.

(ii) *In addition, for $i=2^-,4^-$,*

$$\nu\cdot\rho\ncong\rho,$$

for $\rho=R_\pi^-\boxtimes R_\sigma^-$ a representation of $\mathsf{M}=GSpin_4(\mathfrak{f})\times GL_2(\mathfrak{f})$.

Proof. (i) For $i = 0^+, 2m^+$ if $\Theta = \Theta^+$, $i = 2^-, 4^-$ if $\Theta = \Theta^-$, $i = 1^\dagger, 5^\dagger$ if $\Theta = \Theta^\dagger$, and $i = 0^\ddagger, 2^\ddagger$ if $\Theta = \Theta^\ddagger$, $v[\alpha_i, \Theta]$ acts on the root datum for M, $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$, preserving the set of positive roots Φ^+. So $v[\alpha_i, \Theta]$ gives an automorphism of the based root datum for M

$$\Psi_0 = (X, \Phi^+, X^\vee, (\Phi^+)^\vee).$$

Let B be the Borel subgroup of M given by the positive roots Φ^+, and for each $a \in \Delta$ let $u_a \neq e$ be a fixed element in the root subgroup U_a. By [Sp, Prop 2.13] Aut $\Psi_0(\mathsf{M})$ is isomorphic to the group

$$\mathrm{Aut}(\mathsf{M}, \mathsf{B}, \mathsf{T}, \{u_a\}_{a \in \Delta})$$

of automorphisms of M which stabilize B, T and the set of u_a. Therefore to determine the automorphism of M given by $v[\alpha_i, \Theta]$, we need only find an automorphism of M that stabilizes the set of u_a and gives the same action on T as $v[\alpha_i, \Theta]$.

The action of $v[\alpha_i, \Theta^+]$, $i = 0^+, 2m^+$, on the maximal torus T of $G = GSpin_{4m+5}(k)$ is given by

$$v[\alpha_i, \Theta^+] \cdot (\prod_{j=0}^{2m+2} e_j^*(\lambda_j))$$

$$= e_0^*(\lambda_0 \lambda_1 \ldots \lambda_{2m}) e_1^*(\lambda_{2m}^{-1}) e_2^*(\lambda_{2m-1}^{-1}) \ldots e_{2m}^*(\lambda_1^{-1}) e_{2m+1}^*(\lambda_{2m+1}) \ldots e_{2m+2}^*(\lambda_{2m+2}),$$

for $\lambda_j \in GL_1(k)$. This factors through an action on $\mathsf{T} \subset \mathsf{M}$. If $A \in GL_{2m}(\mathfrak{f})$, $B \in GSpin_5(\mathfrak{f})$ the action of $v[\alpha_i, \Theta^+]$ on M is given by

$$(A, B) \mapsto (w_0({}^tA^{-1})w_0^{-1}, (\det A)B),$$

where $w_0 \in GL_{2m}$ is the matrix with 1 on the antidiagonal and 0 elsewhere.

For $i = 2^-, 4^-$, $v[\alpha_i, \Theta^-]$ acts on the maximal torus T of $G = GSpin_9(k)$ by

$$v[\alpha_i, \Theta^-] \cdot (\prod_{j=0}^{4} e_j^*(\lambda_j)) = e_0^*(\lambda_0\lambda_1\lambda_2) e_1^*(\lambda_2^{-1}) e_2^*(\lambda_1^{-1}) e_3^*(\lambda_3) e_4^*(\lambda_4).$$

Then, if $A \in GL_2(\mathfrak{f})$, $B \in GSpin_4(\mathfrak{f})$ the action of $v[\alpha_i, \Theta^-]$ on M is given by

$$(A, B) \mapsto (w_0({}^tA^{-1})w_0^{-1}, (\det A)B).$$

For $i = 1^\dagger, 5^\dagger$, $v[\alpha_i, \Theta^\dagger]$ acts on the maximal k-split torus S of $G = GSpin_{8,5}(k)$ by

$$v[\alpha_i, \Theta^\dagger] \cdot (\prod_{j=0}^{5} e_j^*(\lambda_j)) = e_0^*(\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4) e_1^*(\lambda_4^{-1}) e_2^*(\lambda_3^{-1}) e_3^*(\lambda_2^{-1}) e_4^*(\lambda_1^{-1}) e_5^*(\lambda_5).$$

This factors through an action on $\mathsf{T} \subset GL_4(\mathfrak{f})$, and by [Sp, Prop 2.13], $v[\alpha_i, \Theta]$ acts on $A \in GL_4(\mathfrak{f})$ as

$$A \mapsto w_0({}^tA^{-1})w_0^{-1}.$$

If $A \in GL_4(\mathfrak{f})$, $B \in {}^2GSpin_4(\mathfrak{f})$ the action of $v[\alpha_i, \Theta^\dagger]$ on M is given by

$$(A, B) \mapsto (w_0({}^tA^{-1})w_0^{-1}, (\det A)B).$$

For $i = 0^\ddagger, 2^\ddagger$, $v[\alpha_i, \Theta^\ddagger]$ acts on the maximal k-split torus S of $G = GSpin_{6,3}(k)$ by

$$v[\alpha_i, \Theta^\ddagger] \cdot (\prod_{j=0}^{3} e_j^*(\lambda_j)) = e_0^*(\lambda_0\lambda_1\lambda_2) e_1^*(\lambda_2^{-1}) e_2^*(\lambda_1^{-1}) e_3^*(\lambda_3).$$

As in the previous case we find, if $A \in GL_2(\mathfrak{f})$, $B \in [({}^2GSpin_2 \times GSpin_3)/\Delta GL_1](\mathfrak{f})$ the action of $v[\alpha_i, \Theta^{\ddagger}]$ on M is given by

$$(A, B) \mapsto (w_0({}^tA^{-1})w_0^{-1}, (\det A)B).$$

By [Bu, 4.1.1], if $R'(A) = R_\sigma({}^tA^{-1})$ then $R' \cong R_\sigma^\vee$. For $i = 0, 2m$ if $\Theta = \Theta^+$, $i = 2^-, 4^-$ if $\Theta = \Theta^-$, $i = 1^\dagger, 5^\dagger$ if $\Theta = \Theta^\dagger$, and $i = 0^\ddagger, 2^\ddagger$ if $\Theta = \Theta^\ddagger$,

$$\rho(v[\alpha_i, \Theta] \cdot (A, B)) = R_\sigma({}^tA^{-1})\omega_{R_\pi}(\det A)R_\pi(B) = R_\sigma^\vee(A)\omega_{R_\pi}(\det A)R_\pi(B).$$

For $m = 1$, as R_π and R_σ are constructed using data from the same TRD parameter $\phi = \phi_1 \oplus \phi_2$, we have $\omega_{R_\pi} = \omega_{R_\sigma}$. As R_σ is a representation of $GL_2(\mathfrak{f})$, by [Bu, 4.1.1],

$$R_\sigma^\vee(\omega_{R_\sigma} \circ \det) \cong R_\sigma.$$

For $m = 2$, both R_π and R_σ are constructed using data from the irreducible TRD parameter ϕ. For

$$\phi(\mathcal{I}_t) = \langle s \rangle \subset GSp_4(\mathbb{C}) \hookrightarrow GL_4(\mathbb{C}),$$

we have

$$s = \mathrm{diag}(\tau, \tau^q, \tau^{q^2}, \tau^{q^3}), \quad \tau\tau^{q^2} = \tau^q\tau^{q^3} =: c.$$

As in Section 4.5, for $R_\pi = R_{\mathsf{T}_\lambda, \chi_\lambda}$, the character χ_λ of $\mathsf{T}_\lambda^{F_\lambda} \subset GSpin_5(\mathfrak{f})$ or ${}^2GSpin_4(\mathfrak{f})$ is represented by s. Similarly, for $R_\pi = R_{\mathsf{T}_\lambda, \chi_\lambda}$, the character χ_λ of $\mathsf{T}_\lambda^{F_\lambda} \subset GL_4(\mathfrak{f})$ is represented by s. We have the character $\omega_\pi \circ \det$ of $GL_4(\mathfrak{f})$ restricted to $\mathsf{T}_\lambda^{F_\lambda}$ is represented by the element

$$\mathrm{diag}(c, c, c, c).$$

Then, as $R_\pi^\vee = R_{\mathsf{T}_\lambda, \chi_\lambda^{-1}}$, by [Ca, 7.2.8] the representation $R_\sigma^\vee \otimes (\omega_\pi \circ \det)$ of $GL_4(\mathfrak{f})$ is represented by the element

$$s^{-1}c.$$

We have

$$s_2s_3s_1s_2 \cdot s^{-1}c = s$$

for $s_2s_3s_1s_2 \in N_{GL_4(\mathbb{C})}(\hat{T})/\hat{T}$ where $s_2s_3s_1s_2$ commutes with the Coxeter element $\hat{w} = s_1s_2s_3 \in N_{GL_4(\mathbb{C})}(\hat{T})/\hat{T}$. Then by [Ca, 7.3.4], $R_\sigma^\vee(\omega_{R_\pi} \circ \det) \cong R_\sigma$.

Therefore, for $i = 0, 2m$ if $\Theta = \Theta^+$, $i = 2^-, 4^-$ if $\Theta = \Theta^-$, $i = 1^\dagger, 5^\dagger$ if $\Theta = \Theta^\dagger$, and $i = 0^\ddagger, 2^\ddagger$ if $\Theta = \Theta^\ddagger$,

$$v[\alpha_i, \Theta] \cdot \rho \cong \rho.$$

(ii) We will now show that the action of ν on $\mathsf{M} = GSpin_4(\mathfrak{f}) \times GL_2(\mathfrak{f})$ does not preserve $\rho = R_\pi^- \boxtimes R_\sigma^-$. We have ν acts on the simple roots in Θ^- by,

$$\nu \cdot \alpha_{0^-} = \alpha_{1^-}, \nu \cdot \alpha_{1^-} = \alpha_{0^-}, \nu \cdot \alpha_{2^-} = \alpha_{2^-}, \nu \cdot \alpha_{3^-} = \alpha_{3^-}, \nu \cdot \alpha_{4^-} = \alpha_{4^-}.$$

Note that ν fixes R_σ^-, but it suffices to show $\nu \cdot R_\pi^- \not\cong R_\pi^-$, which we now do.

Up to conjugation, there is only one minisotropic torus in $GSpin_4(\mathfrak{f})$. So there exists $g \in GSpin_4(\mathfrak{f})$ such that $\nu \cdot \mathsf{T}_\lambda = g\mathsf{T}_\lambda g^{-1}$. Then by [D] there exists a lift $\dot{g}$ of g to $\mathcal{P}$ such that $\nu \cdot T_\lambda = \dot{g}T_\lambda\dot{g}^{-1}$. By replacing ν by $\mathrm{Ad}(\dot{g}^{-1}) \circ \nu$ we can assume ν fixes T_λ. Therefore

$$\nu \in N_{GSpin_5(K)}(T_\lambda)/T_\lambda$$

gives a non-trivial action on T_λ. In fact, since ν induces a non-trivial diagram automorphism for the diagram of $GSpin_4$,

$$\nu \in N_{GSpin_5(K)}(T_\lambda)/T_\lambda \setminus N_{GSpin_4(K)}(T_\lambda)/T_\lambda.$$

Then, since for $R_\pi^- = R_{\mathsf{T}_\lambda,\chi_\lambda}$ the character χ_λ is in general position, by [Ca, 7.3.4]

$$\nu \cdot \rho \ncong \rho.$$

□

Define

$$W(\Theta,\rho) = \{w \in S_\Theta | w\rho \simeq \rho\}.$$

The elements of $W(\Theta,\rho)$ parametrize a basis for $\mathcal{H}(G,\rho)$, with relations given by Theorem (7.12) of [Mo2].

Corollary 6.3.

$$\mathcal{H}(G,\rho) = \langle T_a, T_b, T_c\rangle,$$

$\{a,b\} = \{0^+, 2m^+\}, \{2^-, 4^-\}, \{1^\dagger, 5^\dagger\}$ *or* $\{0^\ddagger, 2^\ddagger\}$, *subject for some* p_i *to the relations*

(i) $T_c T_i = T_i T_c$,

(ii) $T_i^2 = (p_i - 1)T_i + p_i$,

where $i \in \{a,b\}$.

Proof. We have shown

$$W(\Theta,\rho) = \langle v[\alpha_a,\Theta], v[\alpha_b,\Theta], T(e_0^*)\rangle,$$

for $\{a,b\} = \{0^+, 2m^+\}, \{2^-, 4^-\}, \{1^\dagger, 5^\dagger\}$ or $\{0^\ddagger, 2^\ddagger\}$. By [Mo2, Thm 7.12], $\mathcal{H}(G,\rho)$ is generated by three elements T_a, T_b, T_c subject to the given relations. Note that the cocycle μ is trivial in all cases, and in particular μ is trivial when restricted to $\langle T(e_0^*)\rangle$ as $T(e_0^*)$ is a translation. Also, $T_c T_i = T_i T_c$ for all i as $T(e_0^*)v[\alpha_i,\Theta] = v[\alpha_i,\Theta]T(e_0^*)$ (in W) for all i. This is as $wT(e_0^*)w^{-1}(x) = T(w(e_0^*))(x) = T(e_0^*)(x)$ for $x \in \mathcal{A}, w \in W_o$, so all simple reflections commute with $T(e_0^*)$ in W. □

6.2. **A theorem of Lusztig.** In this section, we will compute the parameters p_i in Corollary 6.3. The support of T_i is $\mathcal{P}\dot{w}_i\mathcal{P}$ where $\dot{w}_i \in N_G(T)$ projects to

$$v[\alpha_i,\Theta^+] \in S_{\Theta^+},\ i = 0, 2m, \text{ if } \mathcal{P} = \mathcal{P}^+, \quad v[\alpha_i,\Theta^-] \in S_{\Theta^-},\ i = 2^-, 4^-, \text{ if } \mathcal{P} = \mathcal{P}^-,$$

$$v[\alpha_i,\Theta^\dagger] \in S_{\Theta^\dagger},\ i = 1^\dagger, 5^\dagger, \text{ if } \mathcal{P} = \mathcal{P}^\dagger, \qquad v[\alpha_i,\Theta^\ddagger] \in S_{\Theta^\ddagger},\ i = 0^\ddagger, 2^\ddagger, \text{ if } \mathcal{P} = \mathcal{P}^\ddagger.$$

The elements $\dot{w}_i$ lie in $\mathcal{K}_i$. So T_i is supported in $\mathcal{K}_i$, where i is given as above. Let G_i be the quotient of $\mathcal{K}_i$ by its pro-unipotent radical. Let $\mathsf{P}_i = \mathsf{M}_i\mathsf{N}_i$ be the image of $\mathcal{P}$ in G_i. Recall that $\mathsf{M}_i \cong \mathsf{G}_\lambda^{F_\lambda} \times GL_{2m}(\mathfrak{f})$, so that ρ is a representation of M_i. Consider $V_i = \mathrm{Ind}_{\mathsf{P}_i}^{\mathsf{G}_i}(\rho)$. The algebra of right G_i-endomorphisms of V_i,

$$\mathrm{End}_{\mathsf{G}_i}(V_i) = \mathcal{H}(\mathsf{G}_i,\rho),$$

is isomorphic to the Hecke algebra

$$\mathcal{H}(\mathsf{G}_i,\rho) = \{f : \mathsf{G}_i \longrightarrow \mathrm{End}_{\mathbb{C}}(W^\vee) | f(p_1 g p_2) = \check{\rho}(p_1)f(g)\check{\rho}(p_2),\ p_1, p_2 \in \mathsf{P}_i, g \in \mathsf{G}_i\},$$

where $(\check{\rho}, W^\vee)$ is the contragredient representation of (ρ, W). The finite Hecke algebra $\mathcal{H}(\mathsf{G}_i,\rho)$ can be canonically identified with a subalgebra of $\mathcal{H}(G,\rho)$. This is because $\mathrm{Ind}_{\mathcal{P}}^{\mathcal{K}_i}(\rho) \cong \mathrm{infl}_{\mathsf{G}_i}^{\mathcal{K}_i}(\mathrm{Ind}_{\mathsf{P}_i}^{\mathsf{G}_i}(\rho))$ and $\mathcal{H}(\mathcal{K}_i,\rho)$ can be canonically identified with a subalgebra of $\mathcal{H}(G,\rho)$. By [Mo2, 6.5] $\mathcal{H}(\mathsf{G}_i,\rho)$ is two dimensional. It is generated by a function T_e supported on P_i and a function T_i supported on $\mathsf{P}_i w_i \mathsf{P}_i$. Here, w_i is the image of $\dot{w}_i$ in G_i. We have $w_i \in N_{\mathsf{G}_i}(\mathsf{M}_i) \cap N_{\mathsf{G}_i}(\mathsf{T})$, and $w_i \cdot \rho \cong \rho$. The function T_i satisfies

$$T_i^2 = (p_i - 1)T_i + p_i.$$

Since the endomorphism algebra of V_i is two dimensional it has two irreducible summands:

$$\mathrm{Ind}_{\mathsf{P}_i}^{\mathsf{G}_i}(\rho) = \rho_1 \oplus \rho_2.$$

By [HL], the parameter p_i is the quotient of the degrees of the two irreducible summands.

We will use a theorem of Lusztig, [Lu, Thm 8.6], to compute the parameters p_i.

6.3. **Identification of $\mathsf{G}_i(\bar{\mathfrak{f}})$.** We now describe $\mathsf{G}_i(\bar{\mathfrak{f}})$ in the cases of concern. With the identification of the character group of T with the character group of T, we have the character and cocharacter lattices for T are those for T,

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*,$$

where $n = 2m + 2$. When the simple roots for T are given by Δ^+, the fundamental chamber C^+ in the apartment $\mathcal{A} = X^\vee \otimes \mathbb{R}$ for $G = GSpin_{4m+5}$ is defined by the inequalities

$$1 - e_2 > e_1 > e_2 > \cdots > e_n > 0.$$

When the simple roots for T are given by Δ^-, the fundamental chamber C^- in the apartment $\mathcal{A}$ for $G = GSpin_9$ is defined by the inequalities

$$1 - e_4 > e_3 > e_4 > e_1 > e_2 > 0.$$

When the simple roots for T are given by

$$\Delta^\dagger = \{\beta_{1^\dagger} = e_6 - e_5, \beta_{2^\dagger} = e_5 - e_1, \beta_{3^\dagger} = e_1 - e_2, \beta_{4^\dagger} = e_2 - e_3, \beta_{5^\dagger} = e_3 - e_4, \beta_{6^\dagger} = e_4\},$$

the fundamental chamber $C^\dagger$ in the apartment $\mathcal{A}$ for $G = GSpin_{13}$ is defined by the inequalities

$$1 - e_5 > e_6 > e_5 > e_1 > e_2 > e_3 > e_4 > 0.$$

When the simple roots for T are given by

$$\Delta^\ddagger = \{\beta_{1^\ddagger} = e_4 - e_1, \beta_{2^\ddagger} = e_1 - e_2, \beta_{3^\ddagger} = e_2 - e_3, \beta_{4^\ddagger} = e_3\},$$

the fundamental chamber $C^\ddagger$ in the apartment $\mathcal{A}$ for $G = GSpin_9$ is defined by the inequalities

$$1 - e_1 > e_4 > e_1 > e_2 > e_3 > 0.$$

Let x_i be the vertex of the fundamental chamber C corresponding to the root α_i, (or β_i) in the local Dynkin diagram Π for G. The positive roots of G_i are

$$\Phi_{x_i}^+ = \{a \in \Phi^+ : \langle a, x_i \rangle \in \mathbb{Z}\},$$

where Φ^+ is the set of positive roots for T given by the simple roots above in the various cases. The coroot associated with a root $a \in \Phi_{x_i}^+$ is the same for G_i as for G. We will also list M_i and the dual groups $\mathsf{M}_i^*, \mathsf{G}_i^*$ in each case.

When $\mathcal{P} = \mathcal{P}^+$, the reductive component of reduction mod $\mathfrak{p}$ of $\mathcal{P}$ is

$$\mathsf{M}_{0^+} = \mathsf{M}_{2m^+} = GL_{2m} \times GSpin_5, \qquad \mathsf{M}_{0^+}^* = \mathsf{M}_{2m^+}^* = GL_{2m} \times GSp_4.$$

The root α_{0^+} corresponds to the vertex $x_0 = 0$, so the set of positive roots for G_{0^+} is

$$\Phi_{x_0}^+ = \{e_i \pm e_j \mid 1 \le i < j \le n\} \cup \{e_i \mid 1 \le i \le n\}.$$

Then

$$\mathsf{G}_{0^+} = GSpin_{4m+5}, \qquad \mathsf{G}_{0^+}^* = GSp_{4m+4}.$$

The root α_{2m^+} corresponds to the vertex $x_{2m} = 1/2(e_1^* + e_2^* + \cdots + e_{2m}^*)$, so

$$\Phi_{x_{2m}}^+ = \{e_i \pm e_j | 1 \le i < j \le 2m\} \cup \{e_{2m+1} - e_{2m+2}, e_{2m+2}, e_{2m+1}, e_{2m+1} + e_{2m+2}\}.$$

Therefore,

$$\mathsf{G}_{2m^+} = (GSpin_{4m} \times GSpin_5)/\Delta GL_1, \quad \mathsf{G}^*_{2m^+} = (GSO_{4m} \times GSp_4)^\circ.$$

When $\mathcal{P} = \mathcal{P}^-$,

$$\mathsf{M}_{2^-} = \mathsf{M}_{4^-} = GL_2 \times GSpin_4, \qquad \mathsf{M}^*_{2^-} = \mathsf{M}^*_{4^-} = GL_2 \times GSO_4.$$

The root α_{2^-} corresponds to the vertex $x_{2^-} = 1/2(e_3^* + e_4^*)$, so the set of positive roots for G_{2^-} is

$$\Phi^+_{x_{2^-}} = \{e_1 - e_2, e_2, e_1, e_1 + e_2\} \cup \{e_3 - e_4, e_3 + e_4\}.$$

Then we have

$$\mathsf{G}_{2^-} = (GSpin_5 \times GSpin_4)/\Delta GL_1, \qquad \mathsf{G}^*_{2^-} = (GSp_4 \times GSO_4)^\circ.$$

The root α_{4^-} corresponds to the vertex $x_{4^-} = 1/2(e_1^* + e_2^* + e_3^* + e_4^*)$, so

$$\Phi^+_{x_{4^-}} = \{e_3 \pm e_4, e_4 \pm e_1, e_1 \pm e_2\}.$$

Therefore

$$\mathsf{G}_{4^-} = GSpin_8, \qquad \mathsf{G}^*_{4^-} = GSO_8.$$

When $\mathcal{P} = \mathcal{P}^\dagger$,

$$\mathsf{M}_{1^\dagger}(\bar{\mathfrak{f}}) = \mathsf{M}_{5^\dagger}(\bar{\mathfrak{f}}) = GL_4 \times GSpin_4, \qquad \mathsf{M}^*_{1^\dagger}(\bar{\mathfrak{f}}) = \mathsf{M}^*_{5^\dagger}(\bar{\mathfrak{f}}) = GL_4 \times GSO_4.$$

The root $\beta_{2^\dagger}$ corresponds to the vertex $x_{2^\dagger} = 1/2(e_5^* + e_6^*)$, so the set of positive roots for $\mathsf{G}_{1^\dagger}(\bar{\mathfrak{f}})$ is

$$\Phi^+_{x_{2^\dagger}} = \{e_i \pm e_j |\ 1 \leq i < j \leq 4\} \cup \{e_1, e_2, e_3, e_4\} \cup \{e_6 \pm e_5\}.$$

Then we have

$$\mathsf{G}_{1^\dagger}(\bar{\mathfrak{f}}) = (GSpin_9 \times GSpin_4)/\Delta GL_1, \qquad \mathsf{G}^*_{1^\dagger}(\bar{\mathfrak{f}}) = (GSp_8 \times GSO_4)^\circ.$$

The root $\beta_{6^\dagger}$ corresponds to the vertex $x_{6^\dagger} = 1/2(e_1^* + e_2^* + e_3^* + e_4^* + e_5^* + e_6^*)$. We have

$$\mathsf{G}_{5^\dagger}(\bar{\mathfrak{f}}) = GSpin_{12}, \qquad \mathsf{G}^*_{5^\dagger}(\bar{\mathfrak{f}}) = GSO_{12}.$$

When $\mathcal{P} = \mathcal{P}^\ddagger$,

$$\mathsf{M}_{0^\ddagger}(\bar{\mathfrak{f}}) = \mathsf{M}_{2^\ddagger}(\bar{\mathfrak{f}}) = GL_2 \times (GSpin_2 \times GSpin_3)/\Delta GL_1,$$
$$\mathsf{M}^*_{0^\ddagger}(\bar{\mathfrak{f}}) = \mathsf{M}^*_{2^\ddagger}(\bar{\mathfrak{f}}) = GL_2 \times (GSO_2 \times GSp_2)^\circ.$$

The affine roots $1 - e_4 - e_1$ and $\beta_{1^\ddagger} = e_4 - e_1$ corresponds to the edge of $C^\ddagger$ containing the vertices 0 and $1/2(e_4^*)$, so the set of positive roots for $\mathsf{G}_{0^\ddagger}(\bar{\mathfrak{f}})$ is

$$\{e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, e_1, e_2, e_3\}.$$

Then we have

$$\mathsf{G}_{0^\ddagger}(\bar{\mathfrak{f}}) = (GSpin_7 \times GSpin_2)/\Delta GL_1, \qquad \mathsf{G}^*_{0^\ddagger}(\bar{\mathfrak{f}}) = (GSp_6 \times GSO_2)^\circ.$$

The root $\beta_{3^\ddagger}$ corresponds to the vertex $x_{3^\ddagger} = 1/2(e_4^* + e_1^* + e_2^*)$, so

$$\Phi^+_{x_{3^\ddagger}} = \{e_4 \pm e_1, e_1 \pm e_2, e_4 \pm e_2\} \cup \{e_3\}.$$

Therefore

$$\mathsf{G}_{2^\ddagger}(\bar{\mathfrak{f}}) = (GSpin_3 \times GSpin_6)/\Delta GL_1, \qquad \mathsf{G}^*_{2^\ddagger}(\bar{\mathfrak{f}}) = (GSp_2 \times GSO_6)^\circ.$$

To compute the parameters p_i, [Lu, 8.6] applies only when G_i has a connected center. To use this theorem in the cases where G_i does not have connected center, we will map $\mathsf{G}_i \hookrightarrow \mathsf{G}'_i$

where G_i' has connected center in all cases. Let M_i' be the Levi component of the parabolic subgroup $\mathsf{P}_i' \subset \mathsf{G}_i'$ such that $\mathsf{M}_i \subset \mathsf{M}_i'$ and $\mathsf{P}_i \subset \mathsf{P}_i'$. We list the groups M_i' in the various cases in the following table. The groups $GSpin_{2n}^{\sim}$ are defined in the appendix, where their root datum are also given.

i	$\mathsf{G}_i'(\bar{\mathfrak{f}})$	$\mathsf{M}_i'(\bar{\mathfrak{f}})$
0^+	$GSpin_{4m+5}$	$GL_{2m} \times GSpin_5$
$2m^+$	$(GSpin_{4m}^{\sim} \times GSpin_5)/\Delta GL_1$	$GL_1 \times GL_{2m} \times GSpin_5$
2^-	$(GSpin_5 \times GSpin_4^{\sim})/\Delta GL_1$	$GL_2 \times GSpin_4^{\sim}$
4^-	$GSpin_8^{\sim}$	$GL_2 \times GSpin_4^{\sim}$
$1^\dagger$	$(GSpin_9 \times GSpin_4^{\sim})/\Delta GL_1$	$GL_4 \times GSpin_4^{\sim}$
$5^\dagger$	$GSpin_{12}^{\sim}$	$GL_4 \times GSpin_4^{\sim}$
$0^\ddagger$	$(GSpin_7 \times GSpin_2)/\Delta GL_1$	$GL_2 \times (GSpin_2 \times GSpin_3)/\Delta GL_1$
$2^\ddagger$	$(GSpin_3 \times GSpin_6^{\sim})/\Delta GL_1$	$GL_1 \times GL_2 \times (GSpin_2 \times GSpin_3)/\Delta GL_1$

6.4. **Calculation of parameters p_i.** Let

$$\mathsf{T}_i = \mathsf{T}_{GL_{2m}} \times \mathsf{T}_{\mathsf{G}_\lambda} \subset \mathsf{M}_i,$$

where $\mathsf{T}_{GL_{2m}}$ is the split maximal torus in $GL_{2m}(\bar{\mathfrak{f}})$ and $\mathsf{T}_{\mathsf{G}_\lambda}$ is the split maximal torus in $\mathsf{G}_\lambda(\bar{\mathfrak{f}})$. Recall from Section 4 that there exists an element $\overline{p_\lambda} \in \mathsf{G}_\lambda(\bar{\mathfrak{f}})$ such that $\mathsf{T}_\lambda^{F_\lambda} = \mathrm{Ad}(\overline{p_\lambda})\mathsf{T}^{F_w}$, where w is defined in Section 4.5. Also, from Section 5.1, we have an element $p_{GL_{2m}} \in GL_{2m}(\bar{\mathfrak{f}})$ such that $\mathsf{S}^F = \mathrm{Ad}(p_{GL_{2m}})\mathsf{T}^{F_w}$, where in this case w is a Coxeter element. Denote by

$$\mathsf{S}_i^{F_\lambda} = \mathrm{Ad}(p_{GL_{2m}}, \overline{p_{\lambda(\mathsf{G}_\lambda)}})\mathsf{T}_i^{F_w}$$

where

$$F_\lambda = F_{(GL_{2m})} \otimes F_{\lambda(\mathsf{G}_\lambda)}, \quad F_w = F_{w(GL_{2m})} \otimes F_{w(\mathsf{G}_\lambda)}.$$

We will also denote by F_λ, F_w the twisted Frobenius which acts on G_i' such that the restriction of to G_i is F_λ, F_w. Also, F_λ^*, F_w^* will denote the dual Frobenius which acts on $\mathsf{G}_i'^*$.

The character $\chi_\lambda = \eta_{(GL_{2m})} \otimes \chi_{\lambda(\mathsf{G}_\lambda)}$ of the $F_\lambda = F_{(GL_{2m})} \otimes F_{\lambda(\mathsf{G}_\lambda)}$ stable torus S_i can be viewed as a regular element in a dual F_λ^* stable torus S_i^* of the dual group G_i^*. Let t correspond to $\chi_\lambda = \eta_{(GL_{2m})} \otimes \chi_{\lambda(\mathsf{G}_\lambda)}$. There is a surjective map of dual groups

$$\psi : \mathsf{G}_i'^* \longrightarrow \mathsf{G}_i^*.$$

Let t' be an element of G'^* such that $\psi(t') = t$, and let χ_λ' be the character given by t'. Since χ_λ' is in general position, we have that t' defines an irreducible representation ρ' of $\mathsf{M}_i'(\mathfrak{f})$.

Lemma 6.4. *The restriction of ρ' to $\mathsf{M}_i(\mathfrak{f})$ is isomorphic to ρ.*

Proof. For $i = 0^+, 0^\ddagger$, $\mathsf{M}_i' = \mathsf{M}_i$ so we will not consider these cases. Let $R_{\mathsf{S}_i'}(t')$ be the virtual character (up to a sign) of the representation of $\mathsf{M}_i'^{F_\lambda}$ given by t'. Then $\mathsf{S}_i^{F_\lambda} \subset \mathsf{S}_i'^{F_\lambda}$ where t gives the character of χ_λ of $\mathsf{S}_i^{F_\lambda}$ and t' gives a character χ_λ' of $\mathsf{S}_i'^{F_\lambda}$ such that $\chi_\lambda'(t) = \chi_\lambda(t)$ for

$t \in \mathsf{S}_i^{F_\lambda}$. Let $R|_{\mathsf{M}_i}$ be the restriction of $R_{\mathsf{S}'_i}(t')$ to $\mathsf{M}_i^{F_\lambda}$. By the character formula [Ca, 7.2.8] we have

$$R|_{\mathsf{M}_i}(g) = R_{\mathsf{S}_i}(t)(g), \quad g \in \mathsf{M}_i,$$

so they are the same virtual character. Since $\epsilon_{\mathsf{M}'_i}\epsilon_{\mathsf{S}'_i} = \epsilon_{\mathsf{M}_i}\epsilon_{\mathsf{S}_i}$ we have $\rho'|_{\mathsf{M}_i(\mathfrak{f})} \cong \rho$. □

Lemma 6.5. *The homomorphism $\psi : \mathsf{G}_i'^* \longrightarrow \mathsf{G}_i^*$ central kernel.*

Proof. For $i = 0^+, 0^\ddagger$, $\mathsf{T}'^*_i = \mathsf{T}^*_i$, and for $i = 2m^+, 2^\ddagger$, the kernel of ψ is $\{(g, 1) |\ g \in GL_1\}$ which is in the center of $GL_1 \times \mathsf{M}_i^*$. For the remaining cases it suffices to show the homomorphism $\psi|_{(GSpin^\sim_{2n})^*} : \mathsf{T}'^* \subset (GSpin^\sim_{2n})^* \longrightarrow \mathsf{T}^* \subset (GSpin_{2n})^*$ has kernel contained in the center of $\mathsf{G}_i'^*$. The center of $(GSpin^\sim_{2n})^*$,

$$\{E_{-1}(\mu)E_1(\nu)\dots E_n(\nu)E_0(\nu^2);\ \mu, \nu \in GL_1\},$$

is given by all elements of the split torus T'^*_i that belong to the kernel of all the simple roots. We have

$$\psi|_{(GSpin^\sim_{2n})^*} : \mathsf{T}'^* = \prod_{j=-1}^{n} E_j(\lambda_j) \longrightarrow \mathsf{T}^* = \prod_{j=0}^{n} E_j(\lambda_j), \quad \lambda_j \in GL_1,$$

is given by

$$E_{-1}(\lambda_{-1})E_0(\lambda_0)\dots E_n(\lambda_n) \quad \mapsto \quad E_0(\lambda_0)\dots E_n(\lambda_n),$$

which has kernel contained in the center of $\mathsf{G}_i'^*$. □

Lemma 6.6. *The group $C_{\mathsf{G}_i'^*(\bar{\mathfrak{f}})}(t')$ is connected, reductive with root system*

(i) *type $(A_1)^2$ and Weyl group $W^2_{A_1}$ if $i = 2m^+$ $(m = 2)$, $1^\dagger$;*
(ii) *type $(A_2)^2$ and Weyl group $W^2_{A_2}$ if $i = 0^+$ $(m = 2)$, $5^\dagger$;*
(iii) *type A_1 and Weyl group W_{A_1} if $i = 2m^+$ $(m = 1)$, 2^-, $2^\ddagger$ (case 1), $0^\ddagger$ (case 2);*
(iv) *type A_2 and Weyl group W_{A_2} if $i = 0^+$ $(m = 1)$, 4^-, $0^\ddagger$ (case 1), $2^\ddagger$ (case 2).*

In each case there is one orbit on the simple roots for $W_{\mathsf{G}_i'^*}$, the Weyl group of $C_{\mathsf{G}_i'^*(\bar{\mathfrak{f}})}(t')$, under the action of the dual Frobenius F^*_λ on $\mathsf{G}_i'^*$. In addition, $C_{\mathsf{M}_i'^*(\bar{\mathfrak{f}})}(t') = \mathsf{S}_i'^*$.

Proof. Over $\bar{\mathfrak{f}}$, there exists $x \in \mathsf{M}_i^*$ such that $x\mathsf{S}_i^*x'^{-1} = \mathsf{T}_i^*$. Again over $\bar{\mathfrak{f}}$, there exists $x' \in \mathsf{M}_i'^*$ such that $\psi(x') = x$ and $x'\mathsf{S}_i'^*x'^{-1} = \mathsf{T}_i'^*$. Let $s = x^{-1}tx$ and $s' = x'^{-1}t'x'$, where $\psi(s') = s$. We have an isomorphism of reductive groups

$$\varphi : C_{\mathsf{G}_i'^*}(s') \to C_{\mathsf{G}_i'^*}(t'), \quad \varphi(z) = x'zx'^{-1}.$$

So it suffices to compute $C_{\mathsf{G}_i'^*}(s')$ with its F^*_w structure, which we do in the following.

As the center of G'_i is connected, centralizers of semisimple elements in $\mathsf{G}_i'^*$ are connected. Therefore $C_{\mathsf{G}_i'^*}(s')$ is generated by $\mathsf{T}_i'^*$ and the root groups U_a such that $a(s') = 1$. Since the map

$$\psi : \mathsf{G}_i'^* \to \mathsf{G}_i^*$$

has central kernel, it suffices to compute the root groups U_a such that $a(s) = 1$.

The elements s are constructed from the elements, also called s, listed in the tables in Section 4.5 and Section 5.1. In the case $\rho = R^\ddagger_\sigma \otimes R^\ddagger_\pi$, we have $s = (\mathrm{diag}(\tau_i, \tau_i^q), \mathrm{diag}(\tau_i, \tau_j, \tau_j^q, \tau_i^q))$ or $s = (\mathrm{diag}(\tau_i, \tau_i^q), \mathrm{diag}(\tau_j, \tau_i, \tau_i^q, \tau_j^q))$, $i \neq j \in \{1, 2\}$, is an element of $GL_2 \times (GSO_2(\bar{\mathfrak{f}}) \times GSp_2(\bar{\mathfrak{f}}))^\circ$. Let case 1 and case 2 refer to the situations where

$$\text{case 1} \longleftrightarrow \mathrm{diag}(\tau_i, \tau_i^q) \in GSp_2(\bar{\mathfrak{f}}), \quad \text{case 2} \longleftrightarrow \mathrm{diag}(\tau_i, \tau_i^q) \in GSO_2(\bar{\mathfrak{f}}).$$

Let I be the root system of $W_{\mathsf{G}_i'^*}$, the Weyl group of $C_{\mathsf{G}_i'^*}(s')$. Then in the following table we display the different possibilities.

i	$\mathsf{G}_i^*(\bar{\mathfrak{f}})$	I	$W_{\mathsf{G}_i'^*}$
4^+	$(GSO_8 \times GSp_4)^\circ$	$\{\pm(e_1^*+e_3^*-e_0^*), \pm(e_2^*+e_4^*-e_0^*)\}$	$W_{A_1}^2$
0^+	GSp_{12}	$\langle e_1^*-e_5^*, e_3^*+e_5^*-e_0^*, e_2^*-e_6^*, e_4^*+e_6^*-e_0^*\rangle$	$W_{A_2}^2$
2^+	$(GSO_4 \times GSp_4)^\circ$	$\{\pm(e_1^*+e_2^*-e_0^*)\}$	W_{A_1}
0^+	GSp_8	$\langle e_1^*-e_3^*, e_2^*+e_3^*-e_0^*\rangle$	W_{A_2}
2^-	$(GSp_4 \times GSO_4)^\circ$	$\{\pm(e_1^*+e_2^*-e_0^*)\}$	W_{A_1}
4^-	GSO_8	$\langle e_1^*-e_3^*, e_2^*+e_3^*-e_0^*\rangle$	W_{A_2}
$1^\dagger$	$(GSp_8 \times GSO_4)^\circ$	$\{\pm(e_1^*+e_3^*-e_0^*), \pm(e_2^*+e_4^*-e_0^*)\}$	$W_{A_1}^2$
$5^\dagger$	GSO_{12}	$\langle e_1^*-e_5^*, e_3^*+e_5^*-e_0^*, e_2^*-e_6^*, e_4^*+e_6^*-e_0^*\rangle$	$W_{A_2}^2$
$0^\ddagger$	$(GSp_6 \times GSO_2)^\circ$	case 1, $\langle e_1^*-e_3^*, e_2^*+e_3^*-e_0^*\rangle$	W_{A_2}
$2^\ddagger$	$(GSp_2 \times GSO_6)^\circ$	case 1, $\{\pm(e_1^*+e_2^*-e_0^*)\}$	W_{A_1}
$0^\ddagger$	$(GSp_6 \times GSO_2)^\circ$	case 2, $\{\pm(e_1^*+e_2^*-e_0^*)\}$	W_{A_1}
$2^\ddagger$	$(GSp_2 \times GSO_6)^\circ$	case 2, $\langle e_1^*-e_4^*, e_2^*+e_4^*-e_0^*\rangle$	W_{A_2}

Note that in each case F_w^* acts transitively on the simple roots in I. For $i = 4^+$, F_w^* interchanges $e_1^*+e_3^*-e_0^*$ and $e_2^*+e_4^*-e_0^*$. For $i = 0^+$,

$$F_w^*(e_1^*-e_5^*) = e_4^*+e_6^*-e_0^*, \quad F_w^*(e_3^*+e_5^*-e_0^*) = e_2^*-e_6^*,$$
$$F_w^*(e_2^*-e_6^*) = e_1^*+e_5^*, \quad F_w^*(e_4^*+e_6^*-e_0^*) = e_3^*+e_5^*-e_0^*.$$

For $i = 2^+$, F_w^* fixes $e_1^*+e_2^*-e_0^*$. For $i = 0^+$, F_w^* interchanges $e_1^*-e_3^*$ and $e_2^*+e_3^*-e_0^*$. The other cases are similar.

As χ_λ is in general position, s is not fixed by any element of the Weyl group $(N_{\mathsf{M}_i^*}(\mathsf{T}_i^*)/\mathsf{T}_i^*)^{F_w^*}$ so $C_{\mathsf{M}_i^*}(s) = \mathsf{T}_i^*$. Since $\psi : \mathsf{T}_i'^* \longrightarrow \mathsf{T}_i^*$ has central kernel, we have that $C_{\mathsf{M}_i'^*}(s') = \mathsf{T}_i'^*$ which implies

$$C_{\mathsf{M}_i'^*}(t') = \mathsf{S}_i'^*.$$

□

Lemma 6.7. *We have $p_i' = p_i$.*

Proof. By [Lu, Thm 8.6],

$$\mathrm{End}_{\mathsf{G}_i'(\mathfrak{f})}(\mathrm{Ind}_{\mathsf{P}_i'(\mathfrak{f})}^{\mathsf{G}_i'(\mathfrak{f})}\rho') = \langle T_e, T_i\rangle$$

as an algebra, where T_e is supported on P_i' and T_i satisfies the relation $T_i^2 = (p_i'-1)T_i + p_i'$. Here, T_i corresponds to the unique F_w^*-orbit on I. Therefore, since $\mathrm{End}_{\mathsf{G}_i'}(\mathrm{Ind}_{\mathsf{P}_i'}^{\mathsf{G}_i'}\rho')$ has dimension two, we have that

$$\mathrm{Ind}_{\mathsf{P}_i'}^{\mathsf{G}_i'}\rho' = \rho_1' \oplus \rho_2',$$

where ρ_1' and ρ_2' are distinct irreducible representations of G_i'. As $\rho'|_{\mathsf{M}_i} = \rho$, by Mackey's induction restriction theorem,

$$\mathrm{Ind}_{\mathsf{P}_i'}^{\mathsf{G}_i'}(\rho')|_{\mathsf{G}_i} \cong \mathrm{Ind}_{\mathsf{P}_i}^{\mathsf{G}_i}(\rho) \implies (\rho_1' \oplus \rho_2')|_{\mathsf{G}_i} = \rho_1 \oplus \rho_2.$$

Therefore the quotient of the degrees of ρ_1' and ρ_2' is equal to the quotient of the degrees of ρ_1 and ρ_2, so $p_i' = p_i$. □

Lemma 6.8. *We have:*

(i) $p_i = q^2$ *if* $i = 2m^+ \, (m = 2),\ 1^\dagger$;
(ii) $p_i = q^6$ *if* $i = 0^+ \, (m = 2),\ 5^\dagger$;
(iii) $p_i = q$ *if* $i = 2m^+ \, (m = 1),\ 2^-,\ 2^\ddagger \, (\mathit{case}\, 1),\ 0^\ddagger \, (\mathit{case}\, 2)$;
(iv) $p_i = q^3$ *if* $i = 0^+ \, (m = 1),\ 4^-,\ 0^\ddagger \, (\mathit{case}\, 1),\ 2^\ddagger \, (\mathit{case}\, 2)$.

Proof. Apply formula 8.2.3 of [Lu] to the pair $(W_{A_1}^n, 1)$. This calculation is done in [KM, Appendix], and we have $p_i' = q^n$. Apply formula 8.2.3 of [Lu] to the pair $(W_{A_2}^n, 1)$. This calculation is done in [Sa, §5] and we have $p_i' = q^{3n}$. From the previous lemma, $p_i' = p_i$ in each case. □

Corollary 6.9. *The parameters p_i of $\mathcal{H}(G, \rho)$ are unequal.*

7. Reducibility of generalized principal series

7.1. **A result of Matsumoto.** Let (W, S) be a Coxeter system of type $\tilde{A}_1$, so that W is generated by $S = \{s_1, s_2\}$ where $s_1^2 = s_2^2 = 1$ and $s_1 s_2$ has infinite order. For k a commutative ring and q a real, positive valued, quasi-multiplicative function on W, we denote by $k(W, q)$ the Hecke algebra of type $\tilde{A}_1$ associated to q.

Denote by $q_1 = q(s_1)$, $q_2 = q(s_2)$, and assume $q_2 \geq q_1 \geq 1$. Matsumoto [Ma] defines representations π_ξ of $k(W, q)$, indexed by $\xi \in \mathbb{C}^\times$, such that all irreducible finite dimensional representations of $k(W, q)$ and all irreducible unitary representations of $k(W, q)$ occur in the composition series of such representations. We have:

(i) For $|\xi| = 1$, the representations π_ξ are irreducible and unitary.
(ii) For $\xi \in \mathbb{R}$ such that $1 < \xi \leq \sqrt{q_1 q_2}, -\sqrt{q_2/q_1} \leq \xi < -1$, we have π_ξ is unitary. For $\xi \neq \sqrt{q_1 q_2},\ -\sqrt{q_2/q_1}$, π_ξ is irreducible.
(iii) For $\xi = \sqrt{q_1 q_2},\ -\sqrt{q_2/q_1}$, the composition series of π_ξ is of length two.

Let $\chi_{\xi'}$ denote the irreducible subrepresentation of $\pi_{\sqrt{q_1 q_2}}$ and $\chi_{\xi''}$ the irreducible subrepresentation of $\pi_{-\sqrt{q_2/q_1}}$. Let S^1 be the multiplicative group of complex numbers of modulus 1, and $d\xi$ be the Haar measure on S^1 such that $\int_{S^1} d\xi = 1$. Matsumoto then gives the Plancherel formula for $k(W, q)$. Let $L^1(W, q)$ be the Banach space of L^1 integrable functions $f \in k(W, q)$ with respect to a fixed Haar measure on W. There is a meromorphic function c_1 on $\mathbb{C}^\times$ such that for all $f \in L^1(W, q)$,

$$\begin{aligned} f(e) =& \frac{1}{2} \int_{S^1} \mathrm{Tr}(\pi_\xi(f)) |c_1(\xi)|^{-2} \, d\xi \\ &+ \frac{1 - q_1^{-1} q_2^{-1}}{(1 + q_1^{-1})(1 + q_2^{-1})} \mathrm{Tr}(\chi_{\xi'}(f)) + \frac{1 - q_1 q_2^{-1}}{(1 + q_1)(1 + q_2^{-1})} \mathrm{Tr}(\chi_{\xi''}(f)). \end{aligned}$$

We have

Theorem 7.1. (*Matsumoto*) *A Hecke algebra of type* $\tilde{A}_1$ *has two complementary series if and only if* $q_1 \neq q_2$.

7.2. **Plancherel measures.** To proceed we will need to use information about $\mu(s, \pi \boxtimes \sigma)$, the Plancherel measure associated to the generalized principal series $I(s, \pi \boxtimes \sigma)$. If $P = M \cdot N$ is the maximal parabolic of $GSpin_{4m+5}$ or $GSpin_{2m+4,2m+1}$ defined in Section 5.2, let $\bar{P} = M \cdot \bar{N}$ be the opposite parabolic, and $I_{\bar{P}}(s, \pi \boxtimes \sigma)$ the corresponding generalized principal series representation. There is a local intertwining operator

$$A(s, \pi \boxtimes \sigma, N, \bar{N}) : I(s, \pi \boxtimes \sigma) \longrightarrow I_{\bar{P}}(s, \pi \boxtimes \sigma).$$

The composite $A(s, \pi \boxtimes \sigma, \bar{N}, N) \circ A(s, \pi \boxtimes \sigma, N, \bar{N})$ is a scalar operator on $I(s, \pi \boxtimes \sigma)$ and the Plancherel measure is the meromorphic function defined by

$$\mu(s, \pi \boxtimes \sigma)^{-1} = A(s, \pi \boxtimes \sigma, \bar{N}, N) \circ A(s, \pi \boxtimes \sigma, N, \bar{N}).$$

From [Sil, 5.3-5.4] we have

Proposition 7.2. (*Harish-Chandra*) *If* $\pi \boxtimes \sigma$ *is a unitary supercuspidal representation of* $GSpin_5(k) \times GL_r(k)$ *or* $GSpin_{4,1}(k) \times GL_r(k)$, *then for* s *varying over the real numbers:*

(i) *If* $\sigma \not\cong \sigma^\vee(\omega_\pi \circ \det)$, *then* $\mu(0, \pi \boxtimes \sigma) \neq 0$ *and* $I(s, \pi \boxtimes \sigma)$ *is irreducible for all* $s \in \mathbb{R}$. *In this case only* $I(0, \pi \boxtimes \sigma)$ *is unitary.*

(ii) *If* $\sigma \cong \sigma^\vee(\omega_\pi \circ \det)$, *then there is a unique real* $s_0 \geq 0$ *such that* $I(s_0, \pi \boxtimes \sigma)$ *is reducible. Moreover,* $s_0 > 0$ *if and only if* $\mu(0, \pi \boxtimes \sigma) = 0$, *in which case* s_0 *is the unique pole of* $\mu(s, \pi \boxtimes \sigma)$ *on the positive real axis.*
   (a) *When* $s_0 = 0$, *for all* $s \neq 0$ *we have* $I(s, \pi \boxtimes \sigma)$ *is irreducible and non-unitary. The representation* $I(s_0, \pi \boxtimes \sigma)$ *is of length* 2, *with irreducible subquotients tempered representations.*
   (b) *When* $s_0 > 0$, *we have* $I(s, \pi \boxtimes \sigma)$ *is only reducible for* $s = \pm s_0$. *For* $|s| < |s_0|$, $I(s, \pi \boxtimes \sigma)$ *is unitary. The representation* $I(s_0, \pi \boxtimes \sigma)$ *is of length* 2, *with unique irreducible submodule a discrete series representation.*

Lemma 7.3. *Let* $\pi \boxtimes \sigma$ *be a unitary supercuspidal representation where* π *is a supercuspidal representation of* $GSpin_5(k)$ *or* $GSpin_{4,1}(k)$ *corresponding to a tame regular discrete series* L*-parameter* $\phi = \phi_1 \oplus \cdots \oplus \phi_r$, $r = 1, 2$, *by the construction of DeBacker and Reeder given in Section 4.5. Also* $\sigma \cong \sigma_{\phi_i}$, *where* $1 \leq i \leq r$, *is the depth zero supercuspidal representation of* $GL_{2m}(k)$ *attached to the* L*-parameter* ϕ_i *via the local Langlands correspondence for* GL_{2m}. *Then there are at most two twists* $|\det|^{iv_j}$ *of* σ *such that*

$$\sigma|\det|^{iv_1} \not\cong \sigma|\det|^{iv_2}$$

and such that a representation

$$\mathrm{Ind}_P^G \delta_P^{1/2} \pi \boxtimes \sigma|\det|^{u+iv_j}$$

could reduce for some $u > 0$.

Proof. By Proposition 7.2, $I(s, \pi \boxtimes \sigma|\det|^{iv})$ can reduce for some real $s > 0$ only if

$$\sigma|\det|^{iv} \cong (\sigma|\det|^{iv})^\vee(\omega_\pi \circ \det).$$

Suppose σ_0 satisfies $\sigma_0 \cong \sigma_0^\vee(\omega_\pi \circ \det)$. If $\sigma_0|\det|^{iw}$ satisfies

$$\sigma_0|\det|^{iw} \cong (\sigma_0|\det|^{iw})^\vee(\omega_\pi \circ \det),$$

then

$$\sigma_0|\det|^{2iw} \cong \sigma_0.$$

Then for $g \in Z(GL_{2m})$, $|\det(g)|^{2iw} = 1$ so we must have

$$w = \frac{-k\pi}{2m\log(q)}$$

for some integer k such that $0 \leq k \leq 2$. By the local Langlands conjecture for GL_{2m}, if $\chi : k^\times \to \mathbb{C}^\times$, then

$$\sigma_0(\chi \circ \det) \cong \sigma_0 \iff \phi_{\sigma_0} \otimes \chi \cong \phi_{\sigma_0},$$

where χ is viewed as a character of W_k via local class field theory. We have $\phi_{\sigma_0} = \mathrm{Ind}_{W_{k_{2m}}}^{W_k} \eta$. Then

$$\begin{aligned} \phi_{\sigma_0} \otimes \chi \cong \phi_{\sigma_0} \quad &\iff \quad \mathrm{Ind}_{W_{k_{2m}}}^{W_k}(\eta \cdot \chi|_{W_{k_{2m}}}) \cong \mathrm{Ind}_{W_{k_{2m}}}^{W_k}\eta \\ &\iff \quad \chi|_{W_{k_{2m}}} = 1 \quad \iff \quad \chi^{2m} = 1. \end{aligned}$$

Therefore if $|\cdot|^{iw}$ has order dividing $2m$, then $\sigma_0|\det|^{iw} \cong \sigma_0$. Since $|\cdot|^{iw}$ must have order dividing $4m$, $|\cdot|^{iw}$ gives a nontrivial twist of σ_0 only if it has order $4m$. Any two such twists differ by a character of order dividing $2m$, so there is at most one twist $|\cdot|^{iw}$ of σ_0 such that $\sigma_0|\det|^{iw} \cong (\sigma_0|\det|^{iw})^\vee(\omega_\pi \circ \det)$. □

7.3. Main theorem.

Theorem 7.4. *Let π be a depth zero supercuspidal representation of $GSpin_5(k)$ or $GSpin_{4,1}(k)$ corresponding to a tame regular discrete series L-parameter $\phi = \phi_1 \oplus \cdots \oplus \phi_r$, $r = 1, 2$, by the construction of DeBacker and Reeder given in Section 4.5. Let $\sigma \cong \sigma_{\phi_i}$, where $1 \leq i \leq r$, be the depth zero supercuspidal representation of $GL_{2m}(k)$ attached to the L-parameter ϕ_i via the local Langlands correspondence for GL_{2m}. Then the generalized principal series $I(s, \pi \boxtimes \sigma)$ reduces at a unique $s_0 > 0$.*

Proof. Assume $\pi \boxtimes \sigma$ is unitary. Representations in the Bernstein component $\mathfrak{R}^{[M,\pi\boxtimes\sigma]_G}(G)$ of $I(s, \pi \boxtimes \sigma)$ are parametrized by $\mathcal{H}(G, \rho)$-modules via the map

$$M_\rho : \mathfrak{R}^{[M,\pi\boxtimes\sigma]_G}(G) \to \mathcal{H}(G, \rho) - \mathrm{Mod}, \quad (\kappa, V) \mapsto V_\rho.$$

For an irreducible representation $\kappa = \mathrm{Ind}_P^G \delta_P^{1/2} \pi|\mathrm{sim}|^t \boxtimes \sigma|\det|^s$ in $\mathfrak{R}^{[M,\pi\boxtimes\sigma]_G}(G)$, the function

$$T_c \in \mathcal{H}(G, \rho)$$

acts on $M_\rho(\kappa)$ by the scalar

$$\mathrm{meas}(\mathcal{P})\, \omega_\pi(n)|\mathrm{sim}(n)|^t$$

where $n = e_0^*(\varpi^{-1})$. Therefore irreducible representations in the Bernstein component of $I(s, \pi \boxtimes \sigma)$ which are a composition factor of some

$$\mathrm{Ind}_P^G \delta_P^{1/2} \pi|\mathrm{sim}|^t \boxtimes \sigma|\det|^{u+iv_j}, \quad t = 0,$$

are parametrized by simple modules of a Hecke algebra of type $\tilde{A}_1$. This Hecke algebra of type $\tilde{A}_1$ has unequal parameters by Corollary 6.9 and therefore has to have two complementary series by Theorem 7.1. By Lemma 7.3 (up to isomorphism) there are only two possible v_j such that the representation $\mathrm{Ind}_P^G \delta_P^{1/2} \pi \boxtimes \sigma|\det|^{u+iv_j}$ could reduce for some $u > 0$. In the course of the proof of Lemma 6.2 we showed that for $\rho = R_\pi \boxtimes R_\sigma$,

$$R_\sigma \cong R_\sigma^\vee(\omega_{R_\pi} \circ \det).$$

Then since

$$\sigma = \mathrm{Ind}(\chi_\lambda \otimes R_\sigma),$$

we have

$$\sigma \cong \sigma^\vee(\omega_\pi \circ \det).$$

Therefore, by Proposition 7.2, $I(s, \pi \boxtimes \sigma)$ could reduce for some real $s > 0$. Since there must be two complementary series, $I(s, \pi \boxtimes \sigma)$ does reduce for a unique real $s_0 > 0$. □

8. The L-packets agree

When $\pi \boxtimes \sigma$ is irreducible and generic as a representation of M, we can apply Shahidi's theory of L-functions. Here π is a generic representation of $GSpin_5(k)$. The dual parabolic subgroup of P is $P^\vee = M^\vee \cdot N^\vee \subset GSpin^\vee_{4m+5} = GSp_{4m+4}(\mathbb{C})$, where

$$M^\vee = GSp_4(\mathbb{C}) \times GL_{2m}(\mathbb{C}).$$

Under the adjoint action of $M^\vee$, $\mathfrak{n}^\vee = Lie(N^\vee)$ decomposes as $r_1 \oplus r_2$, where each r_i is a maximal isotypic component for the action of the central torus in $M^\vee$. In this case

$$r_1 = std^\vee \boxtimes std \quad \text{and} \quad r_2 = \mathrm{sim}^{-1} \otimes Sym^2,$$

where std is the standard representation and sim is the similitude character of $GSp_4(\mathbb{C})$.

If $\bar{P}$ is the opposite parabolic and $\bar{P}^\vee$ is the dual of the opposite parabolic, then on the opposite nilpotent radical $\bar{\mathfrak{n}}^\vee$, the adjoint action of $M^\vee$ is the dual representation $r_1^\vee \oplus r_2^\vee$. Shahidi decomposed the Plancherel measure as a product of gamma factors. More precisely [Sh, Thm. 3.5]:

Proposition 8.1. *Suppose that $\pi \boxtimes \sigma$ is a generic representation of $M(k) = GSp_4(k) \times GL_{2m}(k)$. Then*

$$\mu(s, \pi \boxtimes \sigma) = \gamma(s, \pi \boxtimes \sigma, r_1, \psi)\gamma(s, \pi \boxtimes \sigma, r_1^\vee, \psi)\gamma(2s, \pi \boxtimes \sigma, r_2, \psi)\gamma(2s, \pi \boxtimes \sigma, r_2^\vee, \psi).$$

The local factors satisfy

$$\gamma(s, \pi \boxtimes \sigma, r_i, \psi) = \epsilon(s, \pi \times \sigma, r_i, \psi) \cdot \frac{L(1-s, (\pi \times \sigma)^\vee, r_i)}{L(s, \pi \times \sigma, r_i)},$$

where the factors on the right hand side were defined by Shahidi to satisfy the given decomposition of the Plancherel measure.

Collecting [GT, Cor 9.3], [GT, Thm 9.4], and [GTan, §8] we have

Proposition 8.2. *Let π be an irreducible supercuspidal representation of $GSpin_5(k)$ or $GSpin_{4,1}(k)$ with parameter ϕ_π given by the local Langlands conjectures for GSp_4 and its inner form. Then if σ is any irreducible supercuspidal representation of $GL_{2m}(k)$ with parameter ϕ_σ, the Plancherel measure $\mu(s, \pi \boxtimes \sigma)$ is equal to*

$$\gamma(s, \phi_\pi \otimes \phi_\sigma, r_1, \psi)\gamma(s, \phi_\pi \otimes \phi_\sigma, r_1^\vee, \bar{\psi})\gamma(2s, \phi_\pi \otimes \phi_\sigma, r_2, \psi)\gamma(2s, \phi_\pi \otimes \phi_\sigma, r_2^\vee, \bar{\psi}) =$$
$$\gamma(s, \phi_\pi^\vee \otimes \phi_\sigma, \psi)\gamma(-s, \phi_\pi \otimes \phi_\sigma^\vee, \bar{\psi})\gamma(2s, Sym^2\phi_\sigma \otimes \mathrm{sim}\phi_\pi^{-1}, \psi)\gamma(-2s, Sym^2\phi_\sigma^\vee \otimes \mathrm{sim}\phi_\pi, \bar{\psi}).$$

Here

$$\gamma(s, \phi_\pi \otimes \phi_\sigma, r_i, \psi) = \epsilon(s, \phi_\pi \otimes \phi_\sigma, r_i, \psi) \cdot \frac{L(1-s, (\phi_\pi \otimes \phi_\sigma)^\vee, r_i)}{L(s, \phi_\pi \otimes \phi_\sigma, r_i)},$$

are the local factors of Artin type associated to the given representations of the Weil-Deligne group W_k'. For a representation $\phi_1 \otimes \phi_2$ of W_k' the Artin L-function $L(s, \pi_1 \otimes \pi_2)$ is given by

$$L(s, \phi_1 \otimes \phi_2) = \frac{1}{\det(I - q^{-s}(\phi_1 \otimes \phi_2)(\mathrm{Frob})|_{(V_{\phi_1} \otimes V_{\phi_2})^{\mathcal{I}}})}.$$

Lemma 8.3. *Let π be an irreducible supercuspidal representation of $GSpin_5(k)$ or $GSpin_{4,1}(k)$ with L-parameter ϕ_π given by the local Langlands conjectures for GSp_4 and its inner form. Let σ be an irreducible supercuspidal representation of $GL_{2m}(k)$ such that its L-parameter ϕ_σ factors through $GSp_{2m}(\mathbb{C})$ with similitude character $\mathrm{sim}\phi_\pi$. Then*

$$\mu(0, \pi \boxtimes \sigma) = 0 \quad \Longrightarrow \quad \mathrm{Hom}_{W_k}(\phi_\pi, \phi_\sigma) \neq 0.$$

Proof. Let π be a representation of $GSpin_5(k)$ or $GSpin_{4,1}(k)$ with parameter ϕ_π and σ a representation of $GL_{2m}(k)$ with parameter ϕ_σ as in the statement of the lemma. By Proposition 8.2 we have

$$\begin{aligned}\mu(s, \pi \otimes \sigma) =& \gamma(s, \phi_\pi^\vee \otimes \phi_\sigma, \psi)\gamma(-s, \phi_\pi \otimes \phi_\sigma^\vee, \bar{\psi}) \\ & \gamma(2s, Sym^2\phi_\sigma \otimes \mathrm{sim}\phi_\pi^{-1}, \psi)\gamma(-2s, Sym^2\phi_\sigma^\vee \otimes \mathrm{sim}\phi_\pi, \bar{\psi}) \\ =& \epsilon - \text{factors} \cdot \frac{L(1-s, [\phi_\pi^\vee \otimes \phi_\sigma]^\vee)}{L(s, \phi_\pi^\vee \otimes \phi_\sigma)} \cdot \frac{L(1+s, [\phi_\pi \otimes \phi_\sigma^\vee]^\vee)}{L(-s, \phi_\pi \otimes \phi_\sigma^\vee)} \\ & \cdot \frac{L(1-2s, [Sym^2\phi_\sigma \otimes \mathrm{sim}\phi_\pi^{-1}]^\vee)}{L(2s, Sym^2\phi_\sigma \otimes \mathrm{sim}\phi_\pi^{-1})} \cdot \frac{L(1+2s, [Sym^2\phi_\sigma^\vee \otimes \mathrm{sim}\phi_\pi^{-1}]^\vee)}{L(-2s, Sym^2\phi_\sigma^\vee \otimes \mathrm{sim}\phi_\pi)}.\end{aligned}$$

Let $\mu(0, \pi \boxtimes \sigma) = 0$. From the expression for Artin L-functions given above, we can see that none of the numerators has a zero at $s = 0$. We have that ϕ_σ is irreducible and symplectic with similitude character $\mathrm{sim}\phi_\pi$. By Schurs lemma it cannot also be orthogonal with similitude character $\mathrm{sim}\phi_\pi$. Therefore neither $\mathrm{Sym}^2\phi_\sigma \otimes \mathrm{sim}\phi_\pi^{-1}$ or $\mathrm{Sym}^2\phi_\sigma^\vee \otimes \mathrm{sim}\phi_\pi$ can contain a nonzero fixed vector under W_k and neither of the last two denominators has a pole. This forces one of the first two denominators to have a pole. Therefore $\phi_\pi^\vee \otimes \phi_\sigma$ or $\phi_\pi \otimes \phi_\sigma^\vee$ contains the trivial representation and

$$\mathrm{Hom}_{W_k}(\phi_\pi, \phi_\sigma) \neq 0.$$

□

We can now prove:

Theorem 8.4. *Let ϕ be a tame regular discrete series L-parameter. Let L_ϕ^{DR} be the L-packet of depth zero supercuspidal representations of $GSp_4(k)$ or $GU_2(D)$ corresponding to ϕ by the construction of DeBacker and Reeder given in Section 4.5. Let L_ϕ^{GT} be the L-packet of supercuspidal representations of $GSp_4(k)$ or $GU_2(D)$ corresponding to ϕ via the local Langlands conjecture for GSp_4 or $GU_2(D)$. Then*

$$L_\phi^{DR} = L_\phi^{GT}.$$

Proof. In the following assume that all representations π are unitary. Let $\phi = \phi_1 \oplus \cdots \oplus \phi_r$, $r = 1, 2$, be a tame regular discrete series Langlands parameter. Let π_ϕ be a representation of $GSp_4(k)$ or $GU_2(D)$ corresponding to ϕ as in Section 4.5. Under the correspondence defined by Gan and Takeda for GSp_4, or Gan and Tantono for $GU_2(D)$, π_ϕ corresponds to some L-parameter we call ϕ'. Let $\sigma = \sigma_{\phi_i}$, $1 \leq i \leq r$, be the depth zero supercuspidal representation

of GL_{2m} attached to ϕ_i via the local Langlands correspondence for GL_{2m}. Note that if $r = 1$ then $m = 2$, and if $r = 2$ then $m = 1$.

By Theorem 7.4, $I(s, \pi_\phi \boxtimes \sigma)$ reduces for some $s_0 > 0$. By Proposition 7.2 this implies $\mu(0, \pi_\phi \boxtimes \sigma) = 0$. Then, by Lemma 8.3

$$\mathrm{Hom}_{W_k}(\phi_i, \phi') \neq 0.$$

This holds for $1 \leq i \leq r$. Since for $r = 2$, $\phi_1 \ncong \phi_2$,

$$\phi' = \phi$$

in all cases. Therefore $L_\phi^{DR} = L_\phi^{GT}$. □

Corollary 8.5. *The parametrization of DeBacker and Reeder of depth zero supercuspidal representations of $GSp_4(k)$ arising from tame regular discrete series Langlands parameters coincides with the parametrization of Gan and Takeda.*

Proof. Let ϕ be a tame regular discrete series parameter for $GSp_4(k)$. By Lemma 4.3 and Theorem 8.4, the L-packet L_ϕ^{DR} of representations attached to ϕ by DeBacker and Reeder agrees with the L-packet L_ϕ^{GT} given by the local Langlands conjecture for GSp_4. For L-packets of size two, by [GT, Main Thm (ii)] L_ϕ^{GT} contains exactly one generic representation indexed by the trivial character of A_ϕ. By [DR, 6.2.1] the generic representation in L_ϕ^{DR} is also indexed by the trivial character of A_ϕ. Therefore the parametrizations agree. □

Appendix A. Root datum

Here we give a description of the root datum for the various groups appearing in the paper. Let G be a connected reductive linear algebraic group and T a maximal torus of G. Let $X = X^*(T)$ and $X^\vee = X_*(T)$ be the groups of algebraic characters and cocharacters of T. Let Φ and $\Phi^\vee$ be the sets of roots and coroots of T. The quadruple

$$\Psi = (X, \Phi, X^\vee, \Phi^\vee)$$

is the root datum for G. In the following instead of listing the roots Φ and coroots $\Phi^\vee$, we give Δ a set of simple roots for T that generate Φ, and $\Delta^\vee$ a set of simple coroots for T that generate $\Phi^\vee$.

The root datum for GL_n can be given by

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*,$$
$$\Delta = \{a_1 = e_1 - e_2, a_2 = e_2 - e_3, \ldots, a_{n-1} = e_{n-1} - e_n\},$$
$$\Delta^\vee = \{a_1^\vee = e_1^* - e_2^*, a_2^\vee = e_2^* - e_3^*, \ldots, a_{n-1}^\vee = e_{n-1}^* - e_n^*\}.$$

The root datum for $GSpin_{2n+1}$ can be given by [AS, Prop 2.1]

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*,$$
$$\Delta = \{a_1 = e_1 - e_2, a_2 = e_2 - e_3, \ldots, a_{n-1} = e_{n-1} - e_n, a_n = e_n\},$$
$$\Delta^\vee = \{a_1^\vee = e_1^* - e_2^*, a_2^\vee = e_2^* - e_3^*, \ldots, a_{n-1}^\vee = e_{n-1}^* - e_n^*, a_n^\vee = 2e_n^* - e_0^*\}.$$

The root datum for $GSpin_{2n}$ can be given by [AS, Prop 2.1]

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*,$$
$$\Delta = \{a_1 = e_1 - e_2, a_2 = e_2 - e_3, \ldots, a_{n-1} = e_{n-1} - e_n, a_n = e_{n-1} + e_n\},$$
$$\Delta^\vee = \{a_1^\vee = e_1^* - e_2^*, a_2^\vee = e_2^* - e_3^*, \ldots, a_{n-1}^\vee = e_{n-1}^* - e_n^*, a_n^\vee = e_{n-1}^* + e_n^* - e_0^*\}.$$

The root datum for GSp_{2n} can be given by

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*,$$

$$\Delta = \{a_1 = e_1 - e_2, a_2 = e_2 - e_3, \ldots, a_{n-1} = e_{n-1} - e_n, a_n = 2e_n - e_0\},$$
$$\Delta^\vee = \{a_1^\vee = e_1^* - e_2^*, a_2^\vee = e_2^* - e_3^*, \ldots, a_{n-1}^\vee = e_{n-1}^* - e_n^*, a_n^\vee = e_n^*\},$$

The root datum for GSO_{2n} can be given by

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*,$$

$$\Delta = \{a_1 = e_1 - e_2, a_2 = e_2 - e_3, \ldots, a_{n-1} = e_{n-1} - e_n, a_n = e_{n-1} + e_n - e_0\},$$
$$\Delta^\vee = \{a_1^\vee = e_1^* - e_2^*, a_2^\vee = e_2^* - e_3^*, \ldots, a_{n-1}^\vee = e_{n-1}^* - e_n^*, a_n^\vee = e_{n-1}^* + e_n^*\}.$$

Given a quadratic space V, if one has the decomposition $V = V_1 \oplus V_2$, with V_i nondegenerate quadratic subspaces, then $SO(V_1) \times SO(V_2) \subset SO(V)$. If we restrict the covering

$$1 \longrightarrow Z^0 \longrightarrow GSpin(V) \longrightarrow SO(V) \longrightarrow 1$$

to the subgroup $SO(V_1) \times SO(V_2)$ we get

$$1 \longrightarrow Z^0 \longrightarrow GSpin(V_1) \times GSpin(V_2)/\Delta GL_1 \longrightarrow SO(V_1) \times SO(V_2) \longrightarrow 1.$$

Precisely, let

$$(GSpin_{2m} \times GSpin_{2n+1})/\Delta GL_1 = (GSpin_{2m} \times GSpin_{2n+1})/\{h_0^*(\lambda)g_0^*(\lambda) : \lambda \in GL_1\}$$

where h_0^* and g_0^* are given in the following lemma. Let

$$(GSO_{2m} \times GSp_{2n})^\circ = \{(g_1, g_2) \in GSO_{2m} \times GSp_{2n} : \mathrm{sim}(g_1) = \mathrm{sim}(g_2)\}.$$

Lemma A.1. *The root datum for* $(GSpin_{2m} \times GSpin_{2n+1})/\Delta GL_1$ *is given by*

$$X = \mathbb{Z}e_0 \oplus \cdots \oplus \mathbb{Z}e_{m+n}, \quad X^\vee = \mathbb{Z}e_0^* \oplus \cdots \oplus \mathbb{Z}e_{m+n}^*,$$

$$\Delta = \{e_1 - e_2, \ldots, e_{m-1} - e_n, e_{m-1} + e_m\} \cup \{e_{m+1} - e_{m+2}, \ldots, e_{m+n-1} - e_{m+n}, e_{m+n}\},$$

$$\begin{aligned}\Delta^\vee =& \{e_1^* - e_2^*, \ldots, e_{m-1}^* - e_m^*, e_{m-1}^* + e_m^* - e_0^*\} \\ & \cup \{e_{m+1}^* - e_{m+2}^*, \ldots, e_{m+n-1}^* - e_{m+n}^*, 2e_{m+n}^* - e_0^*\}.\end{aligned}$$

Also, $(GSO_{2m} \times GSp_{2n})^\circ = ((GSpin_{2m} \times GSpin_{2n+1})/\Delta GL_1)^\vee$.

Proof. We work with the root datum for $GSpin_{2m}$ and $GSpin_{2n+1}$ given above using the letter h for $GSpin_{2m}$ and g for $GSpin_{2n+1}$. The character lattice for $GSpin_{2m} \times GSpin_{2n+1}$ is the $\mathbb{Z}$-span of

$$h_0, h_1, \ldots, h_m, g_0, g_1, \ldots, g_n.$$

The characters for $G = (GSpin_{2m} \times GSpin_{2n+1})/\Delta GL_1$ are those which are trivial on

$$\{h_0^*(\lambda)g_0^*(\lambda) : \lambda \in GL_1\}.$$

The character lattice for G is the $\mathbb{Z}$-span of

$$h_0 - g_0, h_1, \ldots, h_m, g_1, \ldots, g_n.$$

Using the $\mathbb{Z}$ pairing of the root datum, the cocharacter lattice is the $\mathbb{Z}$-span of

$$\overline{h_0^*} = \overline{g_0^*}, \overline{h_1^*}, \ldots, \overline{h_m^*}, \overline{g_1^*}, \ldots, \overline{g_n^*}.$$

Set

$$e_0 = h_0 - g_0, e_1 = h_1, \ldots, e_m = h_m, e_{m+1} = g_1, e_{m+n} = g_n$$

and

$$e_0^* = \overline{g_0^*}, e_1^* = \overline{h_1^*}, \dots, e_m^* = \overline{h_m^*}, e_{m+1}^* = \overline{g_1^*}, \dots, e_{m+n}^* = \overline{g_n^*}.$$

Using this notation we see that the roots and coroots for G are those given in the statement of the lemma.

Similarly, we work with the root datum for GSO_{2m} given above using the letter h and the root datum for GSp_{2n} given above using the letter g. The characters for $G' = (GSO_{2m} \times GSp_{2n})^\circ$ are equivalence classes of characters for $GSO_{2m} \times GSp_{2n}$. Two characters are equivalent if they have the same value on all elements of G'. The character lattice for G' is the $\mathbb{Z}$-span of

$$\overline{1/2(h_0 + g_0)} = \overline{h_0} = \overline{g_0}, \overline{h_1}, \dots, \overline{h_m}, \overline{g_1}, \dots, \overline{g_n}.$$

Using the $\mathbb{Z}$ pairing of the root datum, the cocharacter lattice is the $\mathbb{Z}$-span of

$$h_0^* + g_0^*, h_1^*, \dots, h_m^*, g_1^*, \dots, g_n^*.$$

Setting

$$e_0 = \overline{1/2(h_0 + g_0)}, e_1 = \overline{h_1}, \dots, e_m = \overline{h_m}, e_{m+1} = \overline{g_1}, \dots, e_{m+n} = \overline{g_n}$$

and

$$e_0^* = h_0^* + g_0^*, e_1^* = h_1^*, \dots, e_m^* = h_m^*, e_{m+1}^* = g_1^*, \dots, e + m + n^* = g_n^*,$$

we see that the roots of G' are the coroots of G, and the coroots of G' are the roots of G. □

The center of $GSpin_{2n}$ is not connected. Let

$$GSpin_{2n}^{\sim} = (GL_1 \times GSpin_{2n})/\{(1,1), (-1, \zeta_0)\},$$

where $\zeta_0 = e_1^*(-1)e_2^*(-1)\dots e_n^*(-1)$ is an element in the center of $GSpin_{2n}$. The root datum for $GSpin_{2n}^{\sim}$ can be given by [AS, 2.6]

$$X = \mathbb{Z}E_{-1} \oplus \mathbb{Z}E_0 \oplus \cdots \oplus \mathbb{Z}E_n, \quad X^\vee = \mathbb{Z}E_{-1}^* \oplus \mathbb{Z}E_0^* \oplus \cdots \oplus \mathbb{Z}E_n^*,$$
$$\Delta = \{E_1 - E_2, \dots, E_{n-1} - E_n, E_{n-1} + E_n - E_{-1}\},$$
$$\Delta^\vee = \{E_1^* - E_2^*, \dots, E_{n-1}^* - E_n^*, E_{n-1}^* + E_n^* - E_0^*\}.$$

The center of $GSpin_{2n}^{\sim}$ is the set of elements which that belong to the kernel of all the simple roots, namely

$$\{E_0^*(\mu)E_1^*(\nu)\dots E_n^*(\nu)E_{-1}^*(\nu^2) : \ \mu, \nu \in GL_1\} \simeq GL_1 \times GL_1,$$

which is connected.

Let

$$(GSpin_{2m}^{\sim} \times GSpin_{2n+1})/\Delta GL_1 = (GSpin_{2m}^{\sim} \times GSpin_{2n+1})/\{E_0^*(\lambda)e_0^*(\lambda) : \ \lambda \in GL_1\}.$$

Lemma A.2. *The root datum for $(GSpin_{2m}^{\sim} \times GSpin_{2n+1})/\Delta GL_1$ is given by*

$$X = \mathbb{Z}E_{-1} \oplus \mathbb{Z}E_0 \oplus \cdots \oplus \mathbb{Z}E_{m+n}, \quad X^\vee = \mathbb{Z}E_{-1}^* \oplus \mathbb{Z}E_0^* \oplus \cdots \oplus \mathbb{Z}E_{m+n}^*,$$

$$\begin{aligned}\Delta =& \{E_1 - E_2, \dots, E_{m-1} - E_n, E_{m-1} + E_m - E_{-1}\} \\ & \cup \{E_{m+1} - E_{m+2}, \dots, E_{m+n-1} - E_{m+n}, E_{m+n}\}, \\ \Delta^\vee =& \{E_1^* - E_2^*, \dots, E_{m-1}^* - E_m^*, E_{m-1}^* + E_m^* - E_0^*\} \\ & \cup \{E_{m+1}^* - E_{m+2}^*, \dots, E_{m+n-1}^* - E_{m+n}^*, 2E_{m+n}^* - E_0^*\}.\end{aligned}$$

Proof. The proof is similar to that of Lemma A.1. □

The center of $(GSpin_{2m}^{\sim} \times GSpin_{2n+1})/\Delta GL_1$ is given by
$$\{E_0^*(\mu)E_1^*(\nu)\dots E_m^*(\nu)E_{-1}^*(\nu^2): \ \mu,\nu \in GL_1\} \simeq GL_1 \times GL_1,$$
which is connected.

References

[AS] M. Asgari and F. Shahidi, *Generic transfer for general spin groups,* Duke Math. J. 132 (2006), no. 1, 137-190.

[Be] J. Bernstein (rédigé par P. Deligne), 'Le 'centre' de Bernstein,' *Représentations des groupes réductifs sur un corps local,* Hermann, Paris, 1984, 1-32.

[BH1] C. Bushnell and G. Henniart, *The essentially tame local Langlands correspondence, I,* J. Amer. Math. Soc. 18 (2005), no. 3, 685-710.

[BH2] C. Bushnell and G. Henniart, *The essentially tame local Langlands correspondence, II: totally ramified representations,* Compos. Math. 141 (2005), no. 4, 979-1011.

[BK1] C. Bushnell and P. Kutzko, *The admissible dual of GL(n) via compact open subgroups,* Annals Math. Studies, Princeton Univ. Press (1993).

[BK2] C. Bushnell and P. Kutzko, *Smooth representations of reductive p-adic groups: structure theory via types,* Proc. London Math. Soc. (3) 77 (1998), no. 3, 582-634.

[Bu] D. Bump, *Automorphic Forms and Representations,* Cambridge Studies in Advanced Mathematics, 55. Cambridge University Press, Cambridge, 1997.

[Ca] R. W. Carter, *Finite Groups of Lie Type. Conjugacy Classes and Complex Characters,* Pure and Applied Mathematics (New York), John Wiley and Sons, New York, 1985.

[CKPS] J. Cogdell, H. Kim, I. Piatetski-Shapiro and F. Shahidi, *Functoriality for the classical groups,* Publ. Math. Inst. Hautes Etudes Sci. No. 99 (2004), 163-233.

[DL] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields,* Ann. of Math., 103 (1976), no. 1, 103-161.

[D] S. DeBacker, *Parametrizing conjugacy classes of maximal unramified tori via Bruhat-Tits theory,* Michigan Math. J. 54 (2006), no. 1, 157-178.

[DR] S. DeBacker and M. Reeder, *Depth-zero supercuspidal L-packets and their stability,* Annals of Math. (2) 169 (2009), no. 3, 795-901.

[Ge] P. Gérardin, *Cuspidal unramified series for central simple algebras over local fields,* Automorphic forms, representations and L-functions, Part 1, pp. 157169, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

[GT] W. T. Gan and S. Takeda, *The local Langlands conjecture for* $GSp(4)$*,* Annals of Math (2) 173 (2011), no. 3, 1841-1882.

[GTan] W. T. Gan and W. Tantono, *The local Langlands conjecture for* $GSp(4)$ *II: The case of inner forms,* submitted.

[He1] G. Henniart, *Correspondence de Langlands-Kazhdan explicite dans le cas non ramifié,* Math. Nach. 158 (1992), 7-26.

[He2] G. Henniart, *Une preuve simple des conjectures de Langlands pour* $GL(n)$ *sur un corps p-adique,* Invent. Math. 139 (2000), 439-455.

[Ho] R. Howlett, *Normalizers of parabolic subgroups of reflection groups,* J. London Math. Soc. (2) 21 (1980), no. 1, 62-80.

[HL] R. Howlett and G. I. Lehrer, *Induced cuspidal representations and generalised Hecke rings,* Invent. Math. 58 (1980), no. 1, 37-64.

[HT] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties,* Annals of Mathematics Studies 151, Princeton, New Jersey, 2001.

[JS] D. Jiang and D. Soudry, *The local converse theorem for* $SO(2n+1)$ *and applications,* Ann. of Math. (2) 157 (2003), no. 3, 743-806.

[Ka] T. Kaletha, *Supercuspidal L-packets via isocrystals,* submitted.

[Ki] J. L. Kim, *Supercuspidal representations: an exhaustion theorem,* J. Amer. Math. Soc. 20 (2007), no. 2, 273-320.

[Ko] R. Kottwitz, *Stable trace formula: cuspidal tempered terms,* Duke Math. J., 51 (1984), 611650.

[KM] P. Kutzko and L. Morris, *Level zero Hecke algebras and parabolic induction: The Siegel case for split classical groups,* IMRN (2006), 1-40.

[Lu] G. Lusztig, *Characters of reductive groups over finite fields,* Annals of Mathematical Studies 107, Princeton University Press, New Jersey, 1984.

[Ma] H. Matsumoto, *Analyse Harmonique dans les Systèmes de Tits Bornologiques de Type Affine,* Lecture Notes in Mathematics 590, Springer Verlag, Heidelberg, 1977.

[Mo1] L. Morris, *Level zero* **G**-*types,* Compositio Mathematica 118 (1999), no. 2, 135-157.

[Mo2] L. Morris, *Tamely ramified Hecke algebras,* Invent. Math. 114 (1993), no. 1, 1-54.

[MP] A. Moy and G. Prasad *Jacquet functors and unrefined minimal K-types,* Comment. Math. Helv. 71 (1996), 98-121.

[Ro] J. Rogawski, *Automorphic representations of unitary groups in three variables,* Annals of Mathematics Studies 123, Princeton University Press, Princeton, NJ, 1990.

[Sa] G. Savin, *Lifting of generic depth zero representations of classical groups,* Journal of Algebra 319 (2008), 3244-3258.

[Sh] F. Shahidi, *A proof of Langlands conjecture on Plancherel measures; complementary series for p-adic groups,* Ann. of Math. (2) 132 (1990), no. 2, 273330.

[Sil] A. Silberger, *Introduction to harmonic analysis on reductive p-adic groups,* Mathematical Notes, 23. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1979.

[Sp] T. A. Springer, *Reductive groups,* Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, 3-27, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

[St] S. Stevens, *The supercuspidal representations of p-adic classical groups,* Invent. Math. 172 (2008), no. 2, 289-352.

[Y] J.-K. Yu, *Construction of tame supercuspidal representations,* J. Amer. Math. Soc. 14 (2001), no. 3, 579-622.

Department of Mathematics, University of California, San Diego, La Jolla, 92093
Current address: Department of Mathematics, University of Iowa, Iowa City, 52242
E-mail address: `jaime-lust@uiowa.edu`